

ANALYTIC ASPECTS OF THE BI-FREE PARTIAL R -TRANSFORM

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ABSTRACT. It is shown that the bi-freely infinitely divisible laws, and only these, can be used to approximate the distributions of sums of identically distributed bi-free pairs of random variables from commuting faces. Furthermore, the necessary and sufficient conditions for this approximation are found. Bi-free convolution semigroups of measures and their Lévy-Khintchine representations are also studied here from an infinitesimal point of view. The proofs relies on the harmonic analysis machinery we developed for integral transforms of two variables, without reference to the combinatorics of moments and bi-free cumulants.

1. INTRODUCTION

The purpose of this paper is to develop a harmonic analysis approach to the partial R -transform and infinitely divisible laws in Voiculescu's bi-free probability theory.

Following [10, 11], given a two-faced pair (a, b) of left variable a and right variable b in a C^* -probability space (\mathcal{A}, φ) , its *bi-free partial R -transform* $R_{(a,b)}$ is defined as the generating series

$$R_{(a,b)}(z, w) = \sum_{m,n \geq 0} R_{m,n}(a, b) z^m w^n$$

of the ordered bi-free cumulants $\{R_{m,n}(a, b) : m, n \geq 0\}$ for the pair (a, b) . As shown by Voiculescu, this partial R -transform actually converges absolutely to the following holomorphic function near the point $(0, 0)$ in \mathbb{C}^2 :

$$(1.1) \quad R_{(a,b)}(z, w) = 1 + zR_a(z) + wR_b(w) - zw/G_{(a,b)}(1/z + R_a(z), 1/w + R_b(w)),$$

where R_a and R_b are respectively the usual R -transforms of a and b , and the function $G_{(a,b)}$ is given by

$$G_{(a,b)}(z, w) = \varphi((zI - a)^{-1}(wI - b)^{-1}).$$

Moreover, if two two-faced pairs (a_1, b_1) and (a_2, b_2) are *bi-free* as in [10], then one has

$$R_{(a_1+a_2, b_1+b_2)}(z, w) = R_{(a_1, b_1)}(z, w) + R_{(a_2, b_2)}(z, w)$$

Date: May 30, 2015; revised on April 28, 2016.

2000 Mathematics Subject Classification. 46L54.

Key words and phrases. Bi-free R -transform; infinitely divisible law; Lévy-Khintchine formula.

for (z, w) near $(0, 0)$.

Since their introduction in 2013, the bi-free R -transform and bi-free cumulants have been the subject of several investigations [3, 6, 7, 8] from the combinatorial perspective. (We also refer the reader to the original papers [10, 11, 12, 13] for the basics of bi-free probability and to [4, 5, 9] for other developments of this theory.) Here in this paper, we would like to contribute to the study of the bi-free R -transform by initiating a new direction which is solely based on the harmonic analysis of integral transforms in two variables. Of course, to accommodate objects like measures or integral transforms, we naturally confine ourselves into the case where all left variables *commute* with all right variables. Thus, the distribution for a two-faced pair of commuting selfadjoint variables is the composition of the expectation functional with the joint spectral measure of these variables, which is a compactly supported Borel probability measure on \mathbb{R}^2 . In particular, the map $G_{(a,b)}$ now becomes the Cauchy transform of the distribution of the pair (a, b) . Furthermore, according to the results in [10], given two compactly supported probabilities μ_1 and μ_2 on \mathbb{R}^2 , one can find two bi-free pairs (a_1, b_1) and (a_2, b_2) of commuting left and right variables such that the law of (a_j, b_j) is the measure μ_j ($j = 1, 2$) and the *bi-free convolution* $\mu_1 \boxplus \mu_2$ of these measures is the distribution of the sum $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$. To reiterate, we now have $R_{\mu_1 \boxplus \mu_2} = R_{\mu_1} + R_{\mu_2}$ near the point $(0, 0)$ in this case.

Under such a framework, we are able to develop a satisfactory theory for bi-free harmonic analysis of probability measures on the plane, and we show that the classical limit theory for infinitely divisible laws, due to Lévy and Khintchine, has a perfect bi-free analogue.

The organization and the description of the results in this paper are as follows. We first begin with continuity results for the two-dimensional Cauchy transform in Section 2. Then we take (1.1) as the new definition for the bi-free R -transform of a planar measure and prove similar continuity results for this transform. In Section 3, we are set to investigate the convergence properties of the scaled bi-free R -transforms $f_n = k_n R_{\mu_n}$, where $k_n \in \mathbb{N}$ and μ_n is a probability law on \mathbb{R}^2 . We find the necessary and sufficient conditions for the pointwise convergence of $\{f_n\}_{n=1}^\infty$, and show that the pointwise limit $f = \lim_{n \rightarrow \infty} f_n$ will be a bi-free R -transform for some probability law ν if the limit f should exist in a certain domain of \mathbb{C}^2 . The class \mathcal{BID} of *bi-freely infinitely divisible laws* is then introduced as the family of all such limit laws ν . Examples are provided and include bi-free analogues of Gaussian and Poisson laws. Other properties of this class such as compound Poisson approximation and a convolution semigroup embedding property are also studied here in Section 3. When applying our results to bi-free

convolution of compactly supported measures, we obtain the criteria for the weak convergence of the measures $\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n$ (k_n times) and the characterization of their limit (namely, being infinitely divisible). Interestingly enough, our limit theorems do not depend on whether the function f_n is a bi-free R -transform or not; that is, the existence of the bi-free convolution for measures with unbounded support does not play a role here. We do, however, show that the binary operation \boxplus can be extended from compactly supported measures to the class \mathcal{BID} . Finally, in Section 4, the bi-free R -transform R_ν of any law $\nu \in \mathcal{BID}$ is studied from a dynamical point of view. We show that R_ν arises as the time derivative of the Cauchy transforms corresponding to the bi-free convolution semigroup $\{\nu_t\}_{t \geq 0}$ generated by the law ν . We then obtain a canonical integral representation for R_ν , called the *bi-free Lévy-Khintchine formula*, from this aspect of the bi-free R -transform.

Acknowledgements. The second named author would like to express his gratitude to Professor Joel Zinn for stimulating conversations that led him to Theorem 3.6 and Proposition 3.11 in this paper. The first-named author was supported through a Coleman Postdoctoral Fellowship at Queen's University and a startup grant for new faculty at National Sun Yat-sen University, and the second-named author was supported by the NSERC Canada Discovery Grant RGPIN-402601. Both authors would like to thank the anonymous referee for his/her review and helpful comments.

2. CONTINUITY THEOREMS

2.1. Two-dimensional Cauchy transforms. We start with some continuity results for the Cauchy transform of two variables. These results are not new and must be known already by harmonic analysts. Since we did not find an appropriate reference for them, we provide their proofs here for the sake of completeness.

Denote by $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ the complex upper half-plane and by \mathbb{C}^- the lower one. For a (positive) planar Borel measure μ satisfying the growth condition

$$(2.1) \quad \int_{\mathbb{R}^2} \frac{1}{\sqrt{1+s^2}\sqrt{1+t^2}} d\mu(s, t) < \infty,$$

the domain of definition for its *Cauchy transform*

$$G_\mu(z, w) = \int_{\mathbb{R}^2} \frac{1}{(z-s)(w-t)} d\mu(s, t)$$

is the set $(\mathbb{C} \setminus \mathbb{R})^2 = \{(z, w) \in \mathbb{C}^2 : z, w \notin \mathbb{R}\}$ consisting of four connected components: $\mathbb{C}^+ \times \mathbb{C}^+$, $\mathbb{C}^- \times \mathbb{C}^+$, $\mathbb{C}^- \times \mathbb{C}^-$, and $\mathbb{C}^+ \times \mathbb{C}^-$. The function G_μ is holomorphic and

satisfies the symmetry

$$G_\mu(z, w) = \overline{G_\mu(\bar{z}, \bar{w})}, \quad (z, w) \in (\mathbb{C} \setminus \mathbb{R})^2.$$

Assume in addition that the Borel measure μ is finite on all compact subsets of \mathbb{R}^2 , so that μ is a σ -finite Radon measure. Since the kernels

$$\frac{1}{\pi^2} \frac{y^2}{(x^2 + y^2)(u^2 + y^2)}, \quad y > 0,$$

form an approximate identity in the space $L^1(\mathbb{R}^2)$ with respect to the Lebesgue measure $dxdu$ on \mathbb{R}^2 , a standard truncation argument and Fubini's theorem imply that for any compactly supported continuous function φ on \mathbb{R}^2 , one has the following inversion formula which recovers the measure μ as a positive linear functional acting on such φ :

$$\int_{\mathbb{R}^2} \varphi d\mu = \lim_{y \rightarrow 0^+} \frac{1}{\pi^2} \int_{\mathbb{R}^2} \varphi(x, u) \left[\underbrace{\int_{\mathbb{R}^2} \frac{y}{(x-s)^2 + y^2} \frac{y}{(u-t)^2 + y^2} d\mu(s, t)}_{=f(x, u, y)} \right] dxdu.$$

Hence the Cauchy transform G_μ determines the underlying measure μ uniquely. Indeed, take the imaginary part, we have

$$\Im \left[\frac{G_\mu(x + iy, u + iy) - G_\mu(x + iy, u - iy)}{2i} \right] = f(x, u, y).$$

Apparently, the definition of G_μ and the above properties can be extended to any Borel signed measure μ whose total variation $|\mu|$ satisfies the growth condition (2.1) and $|\mu|(K) < \infty$ for all compact $K \subset \mathbb{R}^2$.

Let π_1 and π_2 be the projections defined by $\pi_1(s, t) = s$ and $\pi_2(s, t) = t$ for $(s, t) \in \mathbb{R}^2$. For a Borel measure μ on \mathbb{R}^2 , its (principal) *marginal laws* $\mu^{(j)}$ ($j = 1, 2$) are defined as $\mu^{(j)} = \mu \circ \pi_j^{-1}$, the push-forward of μ by these projections. Denoting $\alpha_z = \sqrt{1 + (\Re z / \Im z)^2}$ for any complex number $z \notin \mathbb{R}$, we say that $z \rightarrow \infty$ *non-tangentially* (and write $z \rightarrow_\triangleleft \infty$ to indicate this) if $|z| \rightarrow \infty$ and the quantity α_z remains bounded. The notation $z, w \rightarrow_\triangleleft \infty$ means that both z and w tend to infinity non-tangentially. Our first result says that the one-dimensional Cauchy transform

$$G_{\mu^{(j)}}(z) = \int_{\mathbb{R}} \frac{1}{z - x} d\mu^{(j)}(x), \quad z \notin \mathbb{R},$$

of the marginal law $\mu^{(j)}$ can be recovered from G_μ as a non-tangential limit. Recall that a family \mathcal{F} of finite Borel signed measures on \mathbb{R}^2 is said to be *tight* if

$$\lim_{m \rightarrow \infty} \sup_{\mu \in \mathcal{F}} |\mu|(\mathbb{R}^2 \setminus K_m) = 0,$$

where $K_m = \{(s, t) : |s| \leq m, |t| \leq m\}$. The tightness for measures supported on \mathbb{R} is defined analogously. Since

$$\mathbb{R}^2 \setminus K_m = \{(s, t) : |s| > m, t \in \mathbb{R}\} \cup \{(s, t) : s \in \mathbb{R}, |t| > m\},$$

the finite subadditivity of total variation measure shows that a family \mathcal{F} of Borel signed measures on \mathbb{R}^2 is tight if and only if the collection $\{|\mu|^{(1)}, |\mu|^{(2)} : \mu \in \mathcal{F}\}$ of the marginal laws forms a tight family of Borel signed measures on \mathbb{R} .

Lemma 2.1. *Let \mathcal{F} be a tight family of probability measures on \mathbb{R}^2 . Then for each $(z, w) \in (\mathbb{C} \setminus \mathbb{R})^2$, the limits*

$$\begin{cases} \lim_{\lambda \rightarrow \infty} \lambda G_\mu(z, \lambda) = G_{\mu^{(1)}}(z) \\ \lim_{\lambda \rightarrow \infty} \lambda G_\mu(\lambda, w) = G_{\mu^{(2)}}(w) \end{cases}$$

hold uniformly for $\mu \in \mathcal{F}$. Moreover, these two limits are also uniform for (z, w) in the union $\{(z, w) : |\Im z| \geq \varepsilon > 0, |\Im w| \geq \delta > 0\}$ of polyhalfplanes.

Proof. Observe that for $z, \lambda \notin \mathbb{R}$, $m > 0$, and $\mu \in \mathcal{F}$, we have

$$\begin{aligned} |\lambda G_\mu(z, \lambda) - G_{\mu^{(1)}}(z)| &= \left| \int_{\mathbb{R}^2} \frac{1}{z - s} \left[\frac{\lambda}{\lambda - t} - 1 \right] d\mu(s, t) \right| \\ &\leq \frac{1}{|\Im z|} \int_{\{(s, t) : |t| \leq m\}} \left| \frac{t}{\lambda - t} \right| d\mu(s, t) \\ &\quad + \frac{1}{|\Im z|} \int_{\{(s, t) : |t| > m\}} \left| \frac{\lambda}{\lambda - t} - 1 \right| d\mu(s, t) \\ &\leq \frac{m}{|\Im z| |\Im \lambda|} + \frac{(\alpha_\lambda + 1)}{|\Im z|} \mu(\mathbb{R}^2 \setminus K_m). \end{aligned}$$

Likewise, for $\lambda, w \notin \mathbb{R}$ and $m > 0$, we have

$$|\lambda G_\mu(\lambda, w) - G_{\mu^{(2)}}(w)| \leq \frac{m}{|\Im \lambda| |\Im w|} + \frac{(1 + \alpha_\lambda)}{|\Im w|} \sup_{\mu \in \mathcal{F}} \mu(\mathbb{R}^2 \setminus K_m).$$

The result follows from these estimates. \square

Since $\lim_{z \rightarrow \infty} z G_{\mu^{(1)}}(z) = 1$ uniformly for $\mu^{(1)}$ in any tight family of probability measures on \mathbb{R} [2], we deduce from Lemma 2.1 that

$$(2.2) \quad G_\mu(z, w) = \frac{1}{zw} (1 + o(1)) \quad \text{as } z, w \rightarrow \infty, \quad (z, w) \in (\mathbb{C} \setminus \mathbb{R})^2,$$

uniformly for μ within any tight family of probabilities on \mathbb{R}^2 . This non-tangential limiting behavior plays a role in our next result.

Recall that the set of all finite Borel signed measures on \mathbb{R}^2 is equipped with the topology of weak convergence from duality with continuous and bounded functions on \mathbb{R}^2 under the sup norm. In this topology, a family of signed measures is relatively compact if and only if it is tight and uniformly bounded in total variation norms. Likewise, weak convergence of measures on \mathbb{R} is based on the duality with bounded continuous functions on \mathbb{R} . By Prokhorov's theorem, a tight sequence of probability measures contains a subsequence which converges weakly to a probability measure. We write $\mu_n \Rightarrow \mu$ if the sequence $\{\mu_n\}_{n=1}^\infty$ converges weakly to μ as $n \rightarrow \infty$.

Proposition 2.2. *Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of probability measures on \mathbb{R}^2 . Then the sequence μ_n converges weakly to a probability measure on \mathbb{R}^2 if and only if (i) there exist two open subsets $U \subset \mathbb{C}^+ \times \mathbb{C}^+$ and $V \subset \mathbb{C}^+ \times \mathbb{C}^-$ such that the pointwise limit $\lim_{n \rightarrow \infty} G_{\mu_n}(z, w) = G(z, w)$ exists for every $(z, w) \in U \cup V$ and (ii) $zwG_{\mu_n}(z, w) \rightarrow 1$ uniformly in n as $z, w \rightarrow_\triangleleft \infty$. Moreover, if $\mu_n \Rightarrow \mu$, then we have $G = G_\mu$.*

Proof. Assume $\mu_n \Rightarrow \mu$ for some probability μ on \mathbb{R}^2 . The pointwise convergence $G_{\mu_n} \rightarrow G_\mu$ follows from the definition of weak convergence and the estimate

$$\frac{1}{|z - s||w - t|} \leq \frac{1}{|\Im z||\Im w|}, \quad s, t \in \mathbb{R}; \Im z, \Im w \neq 0.$$

As we have seen earlier, the non-tangential limit (ii) is a consequence of the tightness of $\{\mu_n\}_{n=1}^\infty$.

Conversely, we assume (i) and (ii). The uniform condition (ii) implies that for any given $\varepsilon > 0$, there corresponds $m = m(\varepsilon) > 0$ such that

$$|(iy)(iv)G_{\mu_n}(iy, iv) - 1| < \varepsilon, \quad y, v \geq m, \quad n \geq 1.$$

Taking $v \rightarrow \infty$ and fix $y = m$, Lemma 2.1 shows that

$$\begin{aligned} \varepsilon &\geq \left| (im)G_{\mu_n^{(1)}}(im) - 1 \right| \\ &\geq -\Re \left[(im)G_{\mu_n^{(1)}}(im) - 1 \right] \\ &= \int_{\mathbb{R}} \frac{s^2}{m^2 + s^2} d\mu_n^{(1)}(s) = \int_{\mathbb{R}^2} \frac{s^2}{m^2 + s^2} d\mu_n(s, t) \geq \frac{1}{2} \mu_n(\{(s, t) : |s| > m\}) \end{aligned}$$

for every $n \geq 1$. Similarly, we get

$$\sup_{n \geq 1} \mu_n(\{(s, t) : |t| > m\}) \leq 2\varepsilon$$

after taking $y \rightarrow \infty$ and fixing $v = m$. We conclude from these uniform estimates that the sequence $\{\mu_n\}_{n=1}^\infty$ is tight and hence it possesses weak limit points, at least one of which is a probability law on \mathbb{R}^2 .

We shall argue that there can only be one weak limit for $\{\mu_n\}_{n=1}^\infty$ and therefore the entire sequence μ_n must converge weakly to that unique probability limit law. Indeed, suppose μ and ν are both weak limits of $\{\mu_n\}_{n=1}^\infty$, then the condition (i) implies that

$$G_\mu(z, w) = G(z, w) = G_\nu(z, w), \quad (z, w) \in U \cup V.$$

Since G_μ and G_ν are holomorphic in $(\mathbb{C}^+ \times \mathbb{C}^+) \cup (\mathbb{C}^+ \times \mathbb{C}^-)$ and U and V are open sets, the Identity Theorem in multidimensional complex analysis implies that $G_\mu = G_\nu$ in $(\mathbb{C}^+ \times \mathbb{C}^+) \cup (\mathbb{C}^+ \times \mathbb{C}^-)$. Moreover, we can extend the functional equation $G_\mu = G_\nu$ to the whole $(\mathbb{C} \setminus \mathbb{R})^2$ by taking its complex conjugation and conclude that μ and ν have the same Cauchy transform. Since Cauchy transform determines the underlying measure uniquely, we conclude that $\mu = \nu$, finishing the proof. \square

Note that the condition (ii) in this proposition is in fact equivalent to the tightness of the sequence $\{\mu_n\}_{n=1}^\infty$.

We also have the following continuity result if the limit law has been specified in advance; in which case, the pointwise convergence of Cauchy transforms suffices for the weak convergence of measures.

Proposition 2.3. *Let μ, μ_1, μ_2, \dots be probability measures on \mathbb{R}^2 . Then $\mu_n \Rightarrow \mu$ if and only if $\lim_{n \rightarrow \infty} G_{\mu_n}(z, w) = G_\mu(z, w)$ for every $(z, w) \in (\mathbb{C} \setminus \mathbb{R})^2$.*

Proof. Only the “if” part needs a proof. Assume the pointwise convergence $G_{\mu_n} \rightarrow G_\mu$. Since sets bounded in total variation norm are also weak-star pre-compact sets, the sequence $\{\mu_n\}_{n=1}^\infty$ has a weak-star limit point, say, σ . Observe that for every $(z, w) \in (\mathbb{C} \setminus \mathbb{R})^2$, the corresponding Cauchy kernel

$$\frac{1}{(z-s)(w-t)}, \quad (s, t) \in \mathbb{R}^2,$$

is a continuous function vanishing at infinity. Therefore, being a weak-star limit point of the sequence $\{\mu_n\}_{n=1}^\infty$, the measure σ must satisfy $G_\sigma = G_\mu$ on the domain $(\mathbb{C} \setminus \mathbb{R})^2$. We conclude that any weak-star limit σ is in fact equal to the given probability measure μ and therefore $\mu_n \Rightarrow \mu$ holds. \square

2.2. R -transforms. We now turn to R -transform. Recall that a (truncated) Stolz angle $\Delta_{\alpha, \beta} \subset \mathbb{C}^-$ at zero is the convex domain defined by

$$\Delta_{\alpha, \beta} = \{x + iy \in \mathbb{C}^- : |x| < -\alpha y, y > -\beta\},$$

where $\alpha, \beta > 0$ are two parameters controlling the size of $\Delta_{\alpha, \beta}$. The notation $\overline{\Delta_{\alpha, \beta}}$ means the reflection $\{\bar{z} : z \in \Delta_{\alpha, \beta}\}$. As shown in [2], Stolz angles and their reflections are the natural domains of definition for the usual one-dimensional R -transforms.

Given a Stolz angle $\Delta_{\alpha,\beta}$, we introduce the product domain

$$\Omega_{\alpha,\beta} = (\Delta_{\alpha,\beta} \cup \overline{\Delta_{\alpha,\beta}}) \times (\Delta_{\alpha,\beta} \cup \overline{\Delta_{\alpha,\beta}}) = \{(z, w) : z, w \in \Delta_{\alpha,\beta} \cup \overline{\Delta_{\alpha,\beta}}\}.$$

Since $z \rightarrow 0$ in $\Delta_{\alpha,\beta} \cup \overline{\Delta_{\alpha,\beta}}$ if and only if $1/z \rightarrow_{\triangleleft} \infty$, we have $(z, w) \rightarrow (0, 0)$ within $\Omega_{\alpha,\beta}$ if and only if $1/z, 1/w \rightarrow_{\triangleleft} \infty$. For notational convenience, we will often write Δ for $\Delta_{\alpha,\beta}$ and Ω for $\Omega_{\alpha,\beta}$ in the sequel.

Following Voiculescu [10, 11], given a probability μ on \mathbb{R}^2 , its *bi-free partial R -transform* (or, just R -transform for short) is defined as

$$(2.3) \quad R_{\mu}(z, w) = zR_{\mu(1)}(z) + wR_{\mu(2)}(w) + \left[1 - \frac{1}{h_{\mu}(z, w)}\right],$$

where

$$h_{\mu}(z, w) = G_{\mu}(1/z + R_{\mu(1)}(z), 1/w + R_{\mu(2)}(w)) / zw$$

and the function $R_{\mu(j)}$ is the one-dimensional R -transform for the marginal $\mu^{(j)}$. According to the non-tangential asymptotics (2.2) and the fact that $1/\lambda + R_{\mu(j)}(\lambda) = (1/\lambda)(1 + o(1)) \rightarrow_{\triangleleft} \infty$ as $\lambda \rightarrow 0$ within any Stolz angle at zero [2], there exists a small Stolz angle Δ such that the map h_{μ} is well-defined on the corresponding product domain $\Omega = (\Delta \cup \overline{\Delta}) \times (\Delta \cup \overline{\Delta})$, and the function h_{μ} never vanishes on Ω . Therefore, the resulting bi-free R -transform R_{μ} is well-defined and holomorphic on the set Ω .

It is understood that we will always take such a set Ω as the domain of definition for the R -transform, unless the measure μ is compactly supported. Indeed, if μ has a bounded support, then the domain Ω can be chosen as an open bidisk centered at $(0, 0)$, on which the map R_{μ} admits an absolutely convergent power series expansion with real coefficients. In other words, the bi-free R -transform in this case extends analytically to a neighborhood of $(0, 0)$. Also, the R -transform linearizes the bi-free convolution of compactly supported measures, as shown in Voiculescu's work [10, 11]. In particular, in this case the sum of two such R -transforms is another R -transform on their common domain of definition.

Finally, since the maps $R_{\mu(j)}$ ($j = 1, 2$) satisfy the symmetry property $R_{\mu(j)(\lambda)} = \overline{R_{\mu(j)}(\overline{\lambda})}$ [2], we also have

$$R_{\mu}(z, w) = \overline{R_{\mu}(\overline{z}, \overline{w})}, \quad (z, w) \in \Omega.$$

As in the case of Cauchy transform, the one-dimensional R -transform can be recovered from R_{μ} as a limit.

Lemma 2.4. *Let $R_{\mu} : \Omega \rightarrow \mathbb{C}$ be the R -transform of a probability measure μ on \mathbb{R}^2 . Then:*

(1) For any $(z, w) \in \Omega$, we have

$$\begin{cases} \lim_{\lambda \rightarrow 0} R_\mu(z, \lambda) = zR_{\mu^{(1)}}(z); \\ \lim_{\lambda \rightarrow 0} R_\mu(\lambda, w) = wR_{\mu^{(2)}}(w). \end{cases}$$

(2) $\lim_{(z,w) \rightarrow (0,0)} R_\mu(z, w) = 0$.

Proof. We will only prove the limit $\lim_{\lambda \rightarrow 0} R_\mu(z, \lambda) = zR_{\mu^{(1)}}(z)$, for the second limit follows by the same argument, and (2) is a direct consequence of the non-tangential limit (2.2).

Since $\lim_{\lambda \rightarrow 0} \lambda R_{\mu^{(2)}}(\lambda) = 0$ [2], it suffices to show that

$$h_\mu(z, \lambda) = \frac{G_\mu(1/z + R_{\mu^{(1)}}(z), 1/\lambda + R_{\mu^{(2)}}(\lambda))}{z\lambda} \rightarrow 1$$

for any $z \in \Delta \cup \overline{\Delta}$ as $\lambda \rightarrow 0$. This, however, follows from Lemma 2.1, because $1/\lambda + R_{\mu^{(2)}}(\lambda) = (1/\lambda)(1 + o(1)) \rightarrow_{\prec} \infty$ as $\lambda \rightarrow 0$, $\lambda \in \Delta \cup \overline{\Delta}$. \square

We remark that if we take the tightness of measures into account, then the limits in Lemma 2.4 can also be made uniform over any tight family of probability measures on \mathbb{R}^2 .

An immediate consequence of Lemma 2.4 is that the function R_μ determines the measure μ uniquely.

Proposition 2.5. *If two probability measures μ and ν have the same R -transform, then $\mu = \nu$.*

Proof. If $R_\mu = R_\nu$ on a domain $\Omega = (\Delta \cup \overline{\Delta}) \times (\Delta \cup \overline{\Delta})$, then $\mu^{(j)} = \nu^{(j)}$ ($j = 1, 2$) by Lemma 2.4. The definition (2.3) implies further that

$$G_\mu(1/z + R_{\mu^{(1)}}(z), 1/w + R_{\mu^{(2)}}(w)) = G_\nu(1/z + R_{\nu^{(1)}}(z), 1/w + R_{\nu^{(2)}}(w))$$

for $(z, w) \in \Omega$.

Because the image of the Stolz angle Δ under the map $\lambda \mapsto 1/\lambda + R_{\mu^{(j)}}(\lambda)$ contains a truncated cone

$$\Gamma = \{x + iy \in \mathbb{C} : |x| < ay, y > b\}$$

for some $a, b > 0$ (cf. [2]), we conclude that $G_\mu = G_\nu$ on the open set $(\Gamma \cup \overline{\Gamma}) \times (\Gamma \cup \overline{\Gamma})$. Therefore, we have $G_\mu = G_\nu$ on the entire $(\mathbb{C} \setminus \mathbb{R})^2$ by analyticity. The fact that μ and ν have the same Cauchy transform yields the result. \square

We now present a continuity theorem for the bi-free R -transform.

Proposition 2.6. *Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of probability measures on \mathbb{R}^2 . Then μ_n converges weakly to a probability measure on \mathbb{R}^2 if and only if*

- (1) *there exists a Stolz angle Δ such that all R_{μ_n} are defined in the product domain $\Omega = (\Delta \cup \overline{\Delta}) \times (\Delta \cup \overline{\Delta})$;*
- (2) *the pointwise limit $\lim_{n \rightarrow \infty} R_{\mu_n}(z, w) = R(z, w)$ exists for every (z, w) in the domain Ω ; and*
- (3) *the limit $R_{\mu_n}(-iy, -iv) \rightarrow 0$ holds uniformly in n as $y, v \rightarrow 0^+$.*

Moreover, if $\mu_n \Rightarrow \mu$, then we have $R = R_\mu$.

Proof. Suppose $\mu_n \Rightarrow \mu$. Then we have the weak convergence $\mu_n^{(j)} \Rightarrow \mu^{(j)}$ ($j = 1, 2$) for the marginal laws, because each projection π_j is continuous. By the continuity results for one-dimensional R -transform [2], this marginal weak convergence implies that there exists a Stolz angle Δ such that all $R_{\mu_n^{(j)}}$ ($n \geq 1$) and $R_{\mu^{(j)}}$ are defined in $\Delta \cup \overline{\Delta}$, the pointwise convergence $R_{\mu_n^{(j)}} \rightarrow R_{\mu^{(j)}}$ holds in $\Delta \cup \overline{\Delta}$ as $n \rightarrow \infty$, and $\lambda R_{\mu_n^{(j)}}(\lambda) \rightarrow 0$ uniformly in n as $\lambda \rightarrow 0$ within the set $\Delta \cup \overline{\Delta}$. The last uniform convergence result for $\lambda R_{\mu_n^{(j)}}(\lambda)$ amounts to $1/\lambda + R_{\mu_n^{(j)}}(\lambda) = (1/\lambda)(1 + o(1))$ uniformly in n as $\lambda \rightarrow 0$, $\lambda \in \Delta \cup \overline{\Delta}$. Thus, we conclude that $1/\lambda + R_{\mu_n^{(j)}}(\lambda) \rightarrow_\diamond \infty$ uniformly in n as $\lambda \rightarrow 0$, $\lambda \in \Delta \cup \overline{\Delta}$. By shrinking the Stolz angle Δ if necessary but without changing the notation, Proposition 2.2 (ii) and the definition (2.3) show that all R_{μ_n} are defined on the domain $\Omega = (\Delta \cup \overline{\Delta}) \times (\Delta \cup \overline{\Delta})$, which is the statement (1). The statements (2) and (3) also follow from Proposition 2.2, with the limit function $R = R_\mu$.

Conversely, assume that (1) to (3) hold. The uniform limit condition (3) implies that to each $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ such that

$$|R_{\mu_n}(-iy, -iv)| < \varepsilon, \quad n \geq 1, \quad 0 < y, v < \delta.$$

By taking $v \rightarrow 0$ and fixing y in this inequality, Lemma 2.4 shows that $(-iy)R_{\mu_n^{(1)}}(-iy) \rightarrow 0$ uniformly in n as $y \rightarrow 0^+$. Therefore, again by the results in [2], the sequence $\{\mu_n^{(1)}\}_{n=1}^\infty$ is tight. In the same way, we see that $\{\mu_n^{(2)}\}_{n=1}^\infty$ is also a tight sequence. These two facts together imply the tightness of $\{\mu_n\}_{n=1}^\infty$. By the first part of the proof and Proposition 2.5, any weak limit μ of $\{\mu_n\}_{n=1}^\infty$ will be uniquely determined by the pointwise convergence condition (2). Therefore, the full sequence μ_n must converge weakly to μ . \square

Finally, we remark that the uniform condition (3) in the preceding result is really the tightness of the sequence $\{\mu_n\}_{n=1}^\infty$ in disguise.

3. LIMIT THEOREMS AND INFINITE DIVISIBILITY

We now develop the theory of infinitely divisible R -transforms. Consider an arbitrary sequence $\{\mu_n\}_{n=1}^\infty$ of probabilities on \mathbb{R}^2 , and let k_n be any sequence of positive integers tending to infinity. To motivate our discussion, assume for the moment that each μ_n is compactly supported, then we can view it as the common distribution for the finite sequence $(a_{n1}, b_{n1}), (a_{n2}, b_{n2}), \dots, (a_{nk_n}, b_{nk_n})$ of identically distributed bi-free two-faced pairs of commuting random variables. The theme of our investigation has to do with the following question:

Problem 3.1. What is the class of all possible distributional limits for the sum

$$S_n = (a_{n1}, b_{n1}) + (a_{n2}, b_{n2}) + \dots + (a_{nk_n}, b_{nk_n}) = (\sum_{j=1}^{k_n} a_{nj}, \sum_{j=1}^{k_n} b_{nj}),$$

and what are the conditions for the law of S_n to converge to a specified limit distribution?

Denote by ν_n the distribution of the sum S_n . Note that Voiculescu's R -transform machinery shows that $R_{\nu_n} = k_n R_{\mu_n}$ in a bidisk centered at $(0,0)$. Several necessary conditions for the convergence of $\{\nu_n\}_{n=1}^\infty$ are easy to derive. First, if the sequence ν_n should converge weakly to a probability law ν on \mathbb{R}^2 , then the measures μ_n must satisfy the following *infinitesimality* condition: $\mu_n \Rightarrow \delta_{(0,0)}$. Indeed, Proposition 2.6 shows that there is a universal domain of definition Ω for all R_{ν_n} (and hence for all R_{μ_n}) such that $R_{\mu_n} = R_{\nu_n}/k_n = (R_\nu + o(1)) \cdot o(1)$ as $n \rightarrow \infty$ in Ω and $R_{\mu_n}(-iy, -iv) = R_{\nu_n}(-iy, -iv)/k_n = o(1)$ uniformly in n as $y, v \rightarrow 0^+$. So, we have $\mu_n \Rightarrow \delta_{(0,0)}$. Of course, this also yields $\mu_n^{(j)} = \mu_n \circ \pi_j^{-1} \Rightarrow \delta_0$ ($j = 1, 2$) for the marginal laws. In fact, the converse of this is also true, that is, if the marginal infinitesimality $\mu_n^{(j)} \Rightarrow \delta_0$ holds for $j = 1, 2$, then one has $\mu_n \Rightarrow \delta_{(0,0)}$.

Secondly, to each j , we observe the weak convergence $\nu_n^{(j)} \Rightarrow \nu^{(j)}$ for the marginal laws, and by Lemma 2.4, we have

$$(3.1) \quad \nu_n^{(j)} = \mu_n^{(j)} \boxplus \mu_n^{(j)} \boxplus \dots \boxplus \mu_n^{(j)} \quad (k_n \text{ times}),$$

where \boxplus is the usual free convolution for measures on \mathbb{R} . By the Bercovici-Pata bijection [1], this means that each marginal limit law $\nu^{(j)}$ must be \boxplus -infinitely divisible. On the other hand, by applying the one-dimensional R -transform to the weak convergence $\nu_n^{(j)} \Rightarrow \nu^{(j)}$, we get $\mu_n^{(j)} \Rightarrow \delta_0$.

Thus, assuming the weak convergence of the marginals $\mu_n^{(j)}$ under free convolution, the key to the solution of Problem 3.1 is Theorem 3.2 below, in which the measures μ_n are no longer assumed to be compactly supported.

In order to prove Theorem 3.2, we now review the limit theorems proved in [1] for free convolution \boxplus on \mathbb{R} . The one-dimensional R -transform of an \boxplus -infinitely divisible law ν on \mathbb{R} admits a *free Lévy-Khintchine representation*:

$$R_\nu(z) = \gamma + \int_{\mathbb{R}} \frac{z+x}{1-zx} d\sigma(x), \quad z \notin \mathbb{R},$$

where $\gamma \in \mathbb{R}$ and σ is a finite Borel measure on \mathbb{R} (called *Lévy parameters*). The pair (γ, σ) is unique. Conversely, given Lévy parameters γ and σ , this integral formula determines a unique \boxplus -infinitely divisible law ν on \mathbb{R} . We shall write $\nu = \nu_{\boxplus}^{\gamma, \sigma}$ to indicate this correspondence. In order for the free convolutions (3.1) to converge weakly to $\nu_{\boxplus}^{\gamma, \sigma}$ on \mathbb{R} , it is necessary and sufficient that the limit

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \frac{k_n x}{1+x^2} d\mu_n^{(j)}(x) = \gamma$$

and the one-dimensional weak convergence

$$(3.2) \quad \frac{k_n x^2}{1+x^2} d\mu_n^{(j)}(x) \Rightarrow \sigma$$

hold simultaneously.

Theorem 3.2. *Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of probability measures on \mathbb{R}^2 , and let k_n be a sequence of positive integers such that $\lim_{n \rightarrow \infty} k_n = \infty$. For $j = 1, 2$, assume that $\nu_{jn} = [\mu_n^{(j)}]^{\boxplus k_n}$ converges weakly to $\nu_{\boxplus}^{\gamma_j, \sigma_j}$, the \boxplus -infinitely divisible law determined by Lévy parameters (γ_j, σ_j) . Then the following statements are equivalent:*

- (1) *The pointwise limit $R(z, w) = \lim_{n \rightarrow \infty} k_n R_{\mu_n}(z, w)$ exists for (z, w) in a domain Ω where all R_{μ_n} are defined.*
- (2) *The pointwise limit*

$$D(z, w) = \lim_{n \rightarrow \infty} k_n \int_{\mathbb{R}^2} \frac{z w s t}{(1-zs)(1-wt)} d\mu_n(s, t)$$

exists for any $(z, w) \in (\mathbb{C} \setminus \mathbb{R})^2$.

- (3) *The finite signed measures*

$$d\rho_n(s, t) = k_n \frac{st}{\sqrt{1+s^2}\sqrt{1+t^2}} d\mu_n(s, t)$$

converge weakly to a finite signed measure ρ on \mathbb{R}^2 .

Moreover, if (1), (2), and (3) hold, then the limit function D has a unique integral representation

$$D(z, w) = \int_{\mathbb{R}^2} \frac{zw\sqrt{1+s^2}\sqrt{1+t^2}}{(1-zs)(1-wt)} d\rho(s, t),$$

and we have

$$R(z, w) = zR_{\nu_{\boxplus}^{\gamma_1, \sigma_1}}(z) + wR_{\nu_{\boxplus}^{\gamma_2, \sigma_2}}(w) + D(z, w).$$

In particular, the limit $R(z, w)$ extends analytically to $(\mathbb{C} \setminus \mathbb{R})^2$.

Proof. We have seen that the infinitesimality of $\{\mu_n\}_{n=1}^\infty$ follows from the weak convergence of $\{\nu_{jn}\}_{n=1}^\infty$ ($j = 1, 2$). Let Ω be the universal domain of definition for all R_{μ_n} , whose existence is guaranteed by Proposition 2.6. By the definition (2.3) of R -transform and the assumption $\nu_{jn} \Rightarrow \nu_{\boxplus}^{\gamma_j, \sigma_j}$, the pointwise limit $R(z, w)$ exists at $(z, w) \in \Omega$ if and only if the limit $\lim_{n \rightarrow \infty} k_n[1 - 1/h_{\mu_n}(z, w)]$ does. Moreover, since $\lim_{n \rightarrow \infty} h_{\mu_n}(z, w) = 1$ by Proposition 2.2, we conclude that the limit $R(z, w)$ exists if and only if the limit

$$(3.3) \quad h(z, w) = \lim_{n \rightarrow \infty} k_n[h_{\mu_n}(z, w) - 1]$$

exists in Ω , and in this case we will have

$$R(z, w) = zR_{\nu_{\boxplus}^{\gamma_1, \sigma_1}}(z) + wR_{\nu_{\boxplus}^{\gamma_2, \sigma_2}}(w) + h(z, w).$$

We first show that the limit $h(z, w)$ in (3.3) is in fact equal to the limit $D(z, w)$. For simplicity, we set $G_{jn}^{-1} = G_{\mu_n^{(j)}}^{-1}$ and $R_{jn} = R_{\mu_n^{(j)}}$. According to the identities

$$\frac{1}{1 - zs + zR_{1n}(z)} = \frac{1}{1 - zs} \left[1 - \frac{R_{1n}(z)}{G_{1n}^{-1}(z) - s} \right]$$

and

$$\frac{1}{1 - wt + wR_{2n}(w)} = \frac{1}{1 - wt} \left[1 - \frac{R_{2n}(w)}{G_{2n}^{-1}(w) - t} \right],$$

we can write

$$\begin{aligned} h_{\mu_n}(z, w) - 1 &= -zR_{1n}(z) \int_{\mathbb{R}^2} \frac{1}{(1 - zs)(1 - wt)(zG_{1n}^{-1}(z) - zs)} d\mu_n(s, t) \\ &\quad - wR_{2n}(w) \int_{\mathbb{R}^2} \frac{1}{(1 - zs)(1 - wt)(wG_{2n}^{-1}(w) - wt)} d\mu_n(s, t) \\ &\quad + \int_{\mathbb{R}^2} \frac{zwR_{1n}(z)R_{2n}(w)}{(1 - zs)(1 - wt)(zG_{1n}^{-1}(z) - zs)(wG_{2n}^{-1}(w) - wt)} d\mu_n(s, t) \\ &\quad + \int_{\mathbb{R}^2} \frac{zs + wt - zwt}{(1 - zs)(1 - wt)} d\mu_n(s, t). \end{aligned}$$

As $n \rightarrow \infty$, we observe that $k_n R_{jn} = R_{\nu_{jn}} = R_{\nu_{\boxplus}^{\gamma_j, \sigma_j}} + o(1)$, $R_{jn} = o(1)$, and all the integrals above are of order $(1 + o(1))$ except for the last one. Meanwhile, we have

$$\int_{\mathbb{R}} \frac{s}{1 - zs} d\mu_n^{(1)}(s) = R_{1n}(z) \cdot (1 + o(1))$$

and

$$\int_{\mathbb{R}} \frac{t}{1-wt} d\mu_n^{(2)}(t) = R_{2n}(w) \cdot (1 + o(1))$$

as $n \rightarrow \infty$ (see [1]), as well as the following decomposition

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{zs + wt - zwst}{(1-zs)(1-wt)} d\mu_n(s, t) &= z \int_{\mathbb{R}} \frac{s}{1-zs} d\mu_n^{(1)}(s) + w \int_{\mathbb{R}} \frac{t}{1-wt} d\mu_n^{(2)}(t) \\ &\quad + \int_{\mathbb{R}^2} \frac{zwst}{(1-zs)(1-wt)} d\mu_n(s, t). \end{aligned}$$

We conclude from these findings that

$$k_n[h_{\mu_n}(z, w) - 1] = k_n \int_{\mathbb{R}^2} \frac{zwst}{(1-zs)(1-wt)} d\mu_n(s, t) + o(1)$$

as $n \rightarrow \infty$. It follows that the limit $h(z, w)$ exists if and only if the limit $D(z, w)$ does and $h(z, w) = D(z, w)$, as desired. The equivalence between (1) and (2) is proved, at least for $(z, w) \in \Omega$. At the end of this proof, we shall see that if the limit D exists in Ω then it also exists in the whole space $(\mathbb{C} \setminus \mathbb{R})^2$.

Next, we show that the existence of the limit D in Ω implies (3). Note that the total variation $|\rho_n|$ of the measure ρ_n is given by

$$d|\rho_n|(s, t) = k_n \frac{|st|}{\sqrt{1+s^2}\sqrt{1+t^2}} d\mu_n(s, t).$$

To each $n \geq 1$, we introduce the positive measures

$$d\sigma_n^{(1)}(s) = \frac{k_n s^2}{1+s^2} d\mu_n^{(1)}(s) \quad \text{and} \quad d\sigma_n^{(2)}(t) = \frac{k_n t^2}{1+t^2} d\mu_n^{(2)}(t),$$

on \mathbb{R} and note that the weak convergence condition (3.2) implies that both families $\{\sigma_n^{(j)}\}_{n=1}^{\infty}$ ($j = 1, 2$) are tight and uniformly bounded in total variation norm. Applying the Cauchy-Schwarz inequality to the measure $k_n d\mu_n$, we obtain

$$\begin{aligned} [|\rho_n|(\mathbb{R}^2)]^2 &\leq \int_{\mathbb{R}^2} \frac{s^2}{1+s^2} k_n d\mu_n(s, t) \cdot \int_{\mathbb{R}^2} \frac{t^2}{1+t^2} k_n d\mu_n(s, t) \\ &= \sigma_n^{(1)}(\mathbb{R}) \sigma_n^{(2)}(\mathbb{R}) \leq 2\sigma_1(\mathbb{R}) \sigma_2(\mathbb{R}) < \infty. \end{aligned}$$

Hence, the family $\{\rho_n\}_{n=1}^{\infty}$ is bounded in total variation norm.

On the other hand, for any closed square $K_m = \{(s, t) : |s| \leq m, |t| \leq m\}$, we have

$$(3.4) \quad [|\rho_n|(\mathbb{R}^2 \setminus K_m)]^2 \leq \int_{\mathbb{R}^2 \setminus K_m} \frac{k_n s^2}{1+s^2} d\mu_n(s, t) \cdot \int_{\mathbb{R}^2 \setminus K_m} \frac{k_n t^2}{1+t^2} d\mu_n(s, t)$$

by the Cauchy-Schwarz inequality again. Next, observe that

$$\begin{aligned}
\int_{\{(s,t): |s| \leq m, |t| > m\}} \frac{k_n s^2}{1+s^2} d\mu_n(s, t) &= \int_{\{(s,t): |s| \leq m, |t| > m\}} \frac{s^2(1+t^2)}{(1+s^2)t^2} \frac{k_n t^2}{1+t^2} d\mu_n(s, t) \\
&\leq (1 + 1/m^2) \int_{\{(s,t): |t| > m\}} \frac{k_n t^2}{1+t^2} d\mu_n(s, t) \\
&= (1 + 1/m^2) \sigma_n^{(2)}(\mathbb{R} \setminus [-m, m]).
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_{\mathbb{R}^2 \setminus K_m} \frac{k_n s^2}{1+s^2} d\mu_n(s, t) &= \int_{\{(s,t): |s| > m\}} \frac{k_n s^2}{1+s^2} d\mu_n(s, t) \\
&\quad + \int_{\{(s,t): |s| \leq m, |t| > m\}} \frac{k_n s^2}{1+s^2} d\mu_n(s, t) \\
&\leq \sup_{n \geq 1} \sigma_n^{(1)}(\mathbb{R} \setminus [-m, m]) \\
&\quad + (1 + 1/m^2) \sup_{n \geq 1} \sigma_n^{(2)}(\mathbb{R} \setminus [-m, m]) \\
&\longrightarrow 0
\end{aligned}$$

uniformly in n as $m \rightarrow \infty$, by tightness. Likewise, we also have

$$\lim_{m \rightarrow \infty} \sup_{n \geq 1} \int_{\mathbb{R}^2 \setminus K_m} \frac{k_n t^2}{1+t^2} d\mu_n(s, t) = 0.$$

By virtue of (3.4), these uniform limits imply the tightness of the signed measures $\{\rho_n\}_{n=1}^\infty$. Consequently, the measures $\{\rho_n\}_{n=1}^\infty$ have weak limit points.

Now, suppose that ρ and ρ' are both weak limits for the sequence $\{\rho_n\}_{n=1}^\infty$. We will argue that $\rho = \rho'$ and hence the entire sequence $\{\rho_n\}_{n=1}^\infty$ must converge weakly to ρ . Toward this end we examine the limit D :

$$D(z, w) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \frac{\sqrt{1+s^2}\sqrt{1+t^2}}{(1/z-s)(1/w-t)} d\rho_n(s, t).$$

We deduce from this identity that the signed measures $\sqrt{1+s^2}\sqrt{1+t^2} d\rho(s, t)$ and $\sqrt{1+s^2}\sqrt{1+t^2} d\rho'(s, t)$ have the same Cauchy transforms, first on the open set $\{(z, w) : (1/z, 1/w) \in \Omega\}$ and thus to everywhere by the analyticity of Cauchy transform. This yields $\rho = \rho'$, finishing the proof of (2) implying (3) under the assumption that D exists in Ω . This argument also shows the uniqueness of the integral representation for the function D .

Finally, if (3) holds then the limit D exists not only in Ω but also on the entire space $(\mathbb{C} \setminus \mathbb{R})^2$, because

$$\frac{zw\sqrt{1+s^2}\sqrt{1+t^2}}{(1-zs)(1-wt)}$$

is always a bounded and continuous function in (s, t) , as long as $z, w \notin \mathbb{R}$. This theorem is completely proved. \square

Suppose that the sequences $\{\mu_n\}_{n=1}^\infty$ and $\{k_n\}_{n=1}^\infty$ have an additional property that

$$k_n R_{\mu_n} = R_{\nu_n}, \quad n \geq 1,$$

for some probability law ν_n on \mathbb{R}^2 . For example, this occurs for any k_n when μ_n is compactly supported, and the resulting measure ν_n would be the k_n -th bi-free convolution power of μ_n . In this case it makes sense to investigate the convergence of the laws $\{\nu_n\}_{n=1}^\infty$, and we have the following result which provides an answer to the second part of Problem 3.1. Recall that the signed measures $\{\rho_n\}_{n=1}^\infty$ are defined as in Theorem 3.2 (3).

Corollary 3.3. (Convergence Criteria) *The sequence ν_n converges weakly to a probability law on \mathbb{R}^2 if and only if the marginal free convolutions $[\mu_n^{(j)}]^{\boxplus k_n}$ and the signed measures ρ_n converge weakly on \mathbb{R} and \mathbb{R}^2 , respectively. Furthermore, if $\nu_n \Rightarrow \nu$, $[\mu_n^{(j)}]^{\boxplus k_n} \Rightarrow \nu_{\boxplus}^{\gamma_j, \sigma_j}$ ($j = 1, 2$), and $\rho_n \Rightarrow \rho$, then we have the marginal law $\nu^{(j)} = \nu_{\boxplus}^{\gamma_j, \sigma_j}$ for $j = 1, 2$, and*

$$(3.5) \quad G_\nu(z, w) \left[1 - G_{\sqrt{1+s^2}\sqrt{1+t^2} d\rho} (1/G_{\nu^{(1)}}(z), 1/G_{\nu^{(2)}}(w)) \right] = G_{\nu^{(1)}}(z) G_{\nu^{(2)}}(w)$$

for $(z, w) \in (\mathbb{C} \setminus \mathbb{R})^2$.

Proof. Assume $\nu_n \Rightarrow \nu$ for some probability law ν on \mathbb{R}^2 . Since $\nu_n^{(j)} = [\mu_n^{(j)}]^{\boxplus k_n}$ by Lemma 2.4, we have $[\mu_n^{(j)}]^{\boxplus k_n} \Rightarrow \nu^{(j)}$. It follows that the limit law $\nu^{(j)}$ is \boxplus -infinitely divisible and $\mu_n \Rightarrow \delta_{(0,0)}$. The weak convergence of the measures ρ_n is then a consequence of the pointwise convergence $R_{\nu_n} \rightarrow R_\nu$ by Theorem 3.2.

Conversely, assume the weak convergence of $\{[\mu_n^{(j)}]^{\boxplus k_n}\}_{n=1}^\infty$ and that of $\{\rho_n\}_{n=1}^\infty$. The marginal weak convergence implies that $\{\nu_n\}_{n=1}^\infty$ is tight, and hence the condition (3) of Proposition 2.6 holds for the sequence $\{R_{\nu_n}\}_{n=1}^\infty$. To conclude, we only need to verify the pointwise convergence of $\{R_{\nu_n}\}_{n=1}^\infty$. This property, however, is equivalent to the weak convergence of $\{\rho_n\}_{n=1}^\infty$ by Theorem 3.2. Therefore, $\{\nu_n\}_{n=1}^\infty$ is a weakly convergent sequence.

Finally, the functional equation (3.5) follows from the fact that the R -transform of the limit ν has the integral representation

$$R_\nu(z, w) = zR_{\nu_{\boxplus}^{\gamma_1, \sigma_1}}(z) + wR_{\nu_{\boxplus}^{\gamma_2, \sigma_2}}(w) + G_{\sqrt{1+s^2}\sqrt{1+t^2}d\rho}(1/z, 1/w)$$

for $z, w \notin \mathbb{R}$. □

Next, we present some examples in which the limit laws are constructed via the central limit process or the Poisson type limit theorems. We are mainly interested in probability measures that are *full* in the sense that they are not supported on a one-dimensional line in \mathbb{R}^2 .

Example 3.4. (Bi-free Gaussians) These are the limit laws ν with $\sigma_1 = a\delta_0$, $\sigma_2 = b\delta_0$, and $\rho = c\delta_{(0,0)}$, where $a, b > 0$ and $|c| \leq \sqrt{ab}$. The vector (γ_1, γ_2) represents the mean of the law ν , and

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

is the covariance matrix of ν . The R -transform of ν is given by

$$R_\nu(z, w) = \gamma_1 z + \gamma_2 w + az^2 + bw^2 + czw.$$

The marginal law $\nu^{(1)}$ is the semicircular law with mean γ_1 and variance a , and the law $\nu^{(2)}$ is the same with mean γ_2 and variance b . The equation (3.5) shows that the law ν is compactly supported and absolutely continuous with respect to the Lebesgue measure $ds dt$ on \mathbb{R}^2 , whenever $|c| < \sqrt{ab}$. In the standardized case of $\gamma_1 = \gamma_2 = 0$, $a = b = 1$, and $|c| < 1$, the inversion formula described in Section 2.1 gives the following density formula:

$$d\nu = \frac{1-c^2}{2\pi^2} \frac{\sqrt{4-s^2}\sqrt{4-t^2}}{2(1-c^2)^2 - c(1+c^2)st + 2c^2(s^2+t^2)} ds dt,$$

where $s, t \in [-2, 2]$. In particular, if the marginals are uncorrelated (i.e., $c = 0$), then we have $\nu = \mathcal{S} \otimes \mathcal{S}$, the product of the standard semicircle law \mathcal{S} . The degenerate case $|c| = 1$ corresponds to a non-full probability measure concentrated entirely on a straight line in the plane. Thus, such a degenerate law is purely singular to the Lebesgue measure on \mathbb{R}^2 . The existence of bi-free Gaussian laws is provided by central limit theorems. Indeed, given $a = b = 1$ and a correlation coefficient $c \in [-1, 1]$, let Z_1 and Z_2 be two classically independent real-valued random variables drawn from the same law $(1/2)\delta_{-1} + (1/2)\delta_1$, and let μ_n be the distribution of the random vector

$$(X_n, Y_n) = (\sqrt{(1+c)/2n}Z_1 - \sqrt{(1-c)/2n}Z_2, \sqrt{(1+c)/2n}Z_1 + \sqrt{(1-c)/2n}Z_2).$$

Since the normal domain of attraction of the standard Gaussian law coincides with that of \mathcal{S} , the marginal weak convergence $[\mu_n^{(j)}]^{\boxplus n} \Rightarrow \mathcal{S}$ holds for $j = 1, 2$. On the other hand, the pointwise limit

$$D(z, w) = \lim_{n \rightarrow \infty} n E \left[\frac{zwX_nY_n}{(1 - zX_n)(1 - wY_n)} \right] = czw = G_{c\delta_{(0,0)}}(1/z, 1/w)$$

implies the weak convergence $\rho_n \Rightarrow c\delta_{(0,0)}$. By Corollary 3.3, the measures ν_n converge to the bi-free Gaussian with zero mean and covariance

$$\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}.$$

The general case of a and b follows from a simple rescaling argument.

Example 3.5. (Bi-free compound Poisson laws) Let $\mu \neq \delta_{(0,0)}$ be a probability measure on \mathbb{R}^2 and let $\lambda > 0$ be a given parameter. Consider the sequence

$$\mu_n = (1 - \lambda/n)\delta_{(0,0)} + (\lambda/n)\mu, \quad n \geq 1,$$

and the corresponding marginals $\mu_n^{(j)} = (1 - \lambda/n)\delta_0 + (\lambda/n)\mu^{(j)}$. It is easy to see that the marginal free convolutions $[\mu_n^{(j)}]^{\boxplus n}$ converge weakly to the usual free compound Poisson law with the Lévy parameters

$$\gamma_j = \lambda \int_{\mathbb{R}} \frac{x}{1+x^2} d\mu^{(j)}(x), \quad d\sigma_j(x) = \frac{\lambda x^2}{1+x^2} d\mu^{(j)}(x),$$

and that the signed measures ρ_n converge weakly on \mathbb{R}^2 to

$$\rho = \frac{\lambda st}{\sqrt{1+s^2}\sqrt{1+t^2}} d\mu(s, t).$$

So, by Theorem 3.2, the limit $R = \lim_{n \rightarrow \infty} nR_{\mu_n}$ exists and has the integral representation $R(z, w) = \lambda [(1/zw)G_{\mu}(1/z, 1/w) - 1]$ or, equivalently,

$$R(z, w) = -\lambda + \lambda \int_{\mathbb{R}^2} \frac{1}{(1-zs)(1-wt)} d\mu(s, t), \quad z, w \notin \mathbb{R}.$$

The last integral is in fact the bi-free R -transform of a unique probability distribution $\nu_{\lambda, \mu}$, called the *bi-free compound Poisson law* with rate λ and jump distribution μ , on \mathbb{R}^2 . (The *bi-free Poisson law* with rate λ is defined to be the law $\nu_{\lambda, \delta_{(1,1)}}$.) We now verify the existence of $\nu_{\lambda, \mu}$ by the method of truncation. For each positive integer m , let f_m be a compactly supported, continuous function such that $0 \leq f_m \leq 1$, $f_m(s, t) = 1$ for $(s, t) \in K_m = \{(s, t) : |s| \leq m, |t| \leq m\}$, and f_m vanishes on the complement $\mathbb{R}^2 \setminus K_{m+1}$. For sufficiently large m , we introduce the following truncation

$$d\mu^m = c_m f_m d\mu,$$

where the normalization constant c_m is chosen so that $\mu^m(\mathbb{R}^2) = 1$. We observe from an application of the dominated convergence theorem that $c_m \rightarrow 1$ and the weak convergence $\mu^m \Rightarrow \mu$ on \mathbb{R}^2 as $m \rightarrow \infty$. After applying the above limiting process to each truncation μ^m , Corollary 3.3 provides a limit law ν_{λ, μ^m} whose R -transform is given by

$$R_{\nu_{\lambda, \mu^m}}(z, w) = \lambda [(1/zw)G_{\mu^m}(1/z, 1/w) - 1], \quad z, w \notin \mathbb{R}.$$

By Proposition 2.2, the weak convergence $\mu^m \Rightarrow \mu$ implies that $\{R_{\nu_{\lambda, \mu^m}}\}_{m=1}^\infty$ converges pointwisely to the integral R and $R_{\nu_{\lambda, \mu^m}}(-iy, -iv) \rightarrow 0$ uniformly in m as $y, v \rightarrow 0^+$. Therefore, by Proposition 2.6, these conditions show further that the laws ν_{λ, μ^m} converge weakly to a unique limit distribution $\nu_{\lambda, \mu}$ and $R = R_{\nu_{\lambda, \mu}}$, as desired.

We continue our investigation on the characterization of the limit laws. Let $R = \lim_{n \rightarrow \infty} k_n R_{\mu_n}$ be the pointwise limit in Theorem 3.2. Then for any integer $m \geq 2$, the function R/m is the limit of $[k_n/m]R_{\mu_n}$ where $[x]$ indicates the integral part of $x \in \mathbb{R}$. In other words, the limit of $k_n R_{\mu_n}$ can always be decomposed into a sum of m identical functions of the same kind. This nature of the limit R can also be seen from its integral representation, say,

$$R(z, w)/m = zR_{\nu_{\frac{\gamma_1}{m}, \sigma_1/m}}(z) + wR_{\nu_{\frac{\gamma_2}{m}, \sigma_2/m}}(w) + G_{\sqrt{1+s^2}\sqrt{1+t^2}d\rho/m}(1/z, 1/w),$$

where the quintuple $(\gamma_1, \gamma_2, \sigma_1, \sigma_2, \rho)$ of numbers and measures are provided by Theorem 3.2. In the next result we show that every limiting integral form of this kind is indeed a bi-free R -transform.

Theorem 3.6. *Let $\{\mu_n\}_{n=1}^\infty$ and $\{k_n\}_{n=1}^\infty$ be two sequences of probability measures and positive integers satisfying the hypotheses of Theorem 3.2, and assume that the limit function $R = \lim_{n \rightarrow \infty} k_n R_{\mu_n}$ exists in a domain Ω . Then there exists a unique probability law ν on \mathbb{R}^2 such that $R = R_\nu$ on Ω .*

Proof. The uniqueness statement follows from Proposition 2.5; we shall prove the existence of ν . The idea of the proof is to consider a two-dimensional “lifting” of the integral representation of the limit R . First, from the proof of Theorem 3.2 we see that both

$$\left\{ \sigma_{1n} = \frac{k_n s^2}{1+s^2} d\mu_n(s, t) : n \geq 1 \right\} \quad \text{and} \quad \left\{ \sigma_{2n} = \frac{k_n t^2}{1+t^2} d\mu_n(s, t) : n \geq 1 \right\}$$

are tight families of Borel measures on \mathbb{R}^2 . By dropping to subsequences but without changing the notations, we may and do assume that they are both weakly convergent sequences of measures on \mathbb{R}^2 . We write

$$\gamma_{1n} = \int_{\mathbb{R}^2} \frac{k_n s}{1+s^2} d\mu_n(s, t) \quad \text{and} \quad \gamma_{2n} = \int_{\mathbb{R}^2} \frac{k_n t}{1+t^2} d\mu_n(s, t)$$

so that $\lim_{n \rightarrow \infty} \gamma_{jn} = \gamma_j$ for $j = 1, 2$. Note that we have

$$R_{\nu_{\boxplus}^{\gamma_1, \sigma_1}}(z) = \lim_{n \rightarrow \infty} \left[\gamma_{1n} + \int_{\mathbb{R}^2} \frac{z+s}{1-zs} d\sigma_{1n}(s, t) \right]$$

and a similar formula for the function $R_{\nu_{\boxplus}^{\gamma_2, \sigma_2}}$. Together with the weak convergence $\rho_n \Rightarrow \rho$, a straightforward calculation leads to the following asymptotic Poissonization for the limit R :

$$\begin{aligned} R(z, w) &= zR_{\nu_{\boxplus}^{\gamma_1, \sigma_1}}(z) + wR_{\nu_{\boxplus}^{\gamma_2, \sigma_2}}(w) + \lim_{n \rightarrow \infty} k_n \int_{\mathbb{R}^2} \frac{zwt}{(1-zs)(1-wt)} d\mu_n(s, t) \\ &= \lim_{n \rightarrow \infty} k_n [(1/zw)G_{\mu_n}(1/z, 1/w) - 1] \\ &= \lim_{n \rightarrow \infty} R_{\nu_{k_n, \mu_n}}(z, w) \end{aligned}$$

for $z, w \notin \mathbb{R}$.

The rest of the proof will be devoted to showing the tightness of the compound Poisson laws $\{\nu_{k_n, \mu_n}\}_{n=1}^{\infty}$; that is, $R_{\nu_{k_n, \mu_n}}(-iy, -iv) \rightarrow 0$ uniformly in n as $y, v \rightarrow 0^+$. Indeed, if this were true then the limit ν of $\{\nu_{k_n, \mu_n}\}_{n=1}^{\infty}$ would satisfy $R = R_{\nu}$ by Proposition 2.6, bringing us to the end of this proof. To this purpose, consider first the estimates

$$\left| \frac{(-iy)^2 - iys}{1 + iys} \right| \leq 1, \quad s \in \mathbb{R},$$

and

$$\left| \frac{-iy + s}{1 + iys} \right| \leq \sqrt{2} \frac{y + |s|}{1 + y|s|} \leq \sqrt{2}(1 + T), \quad |s| \leq T,$$

for arbitrary $0 < y \leq 1$ and $T > 0$. These observations imply that

$$\sup_{n \geq 1} \left| \int_{\mathbb{R}^2} \frac{(-iy)^2 - iys}{1 + iys} d\sigma_{1n}(s, t) \right| \leq \sqrt{2}(1 + T)y + \sup_{n \geq 1} \sigma_{1n}(\{(s, t) : |s| > T\})$$

for these y and T . Since the tail-sums of the tight sequence $\{\sigma_{1n}\}_{n=1}^{\infty}$ can be made uniformly small as we wish and since the sequence $\{\gamma_{1n}\}_{n=1}^{\infty}$ is bounded, we conclude that

$$\lim_{y \rightarrow 0^+} \sup_{n \geq 1} \left| i\gamma_{1n}y + \int_{\mathbb{R}^2} \frac{(-iy)^2 - iys}{1 + iys} d\sigma_{1n}(s, t) \right| = 0.$$

Using the same method, it can be shown that

$$\lim_{v \rightarrow 0^+} \sup_{n \geq 1} \left| i\gamma_{2n}v + \int_{\mathbb{R}^2} \frac{(-iv)^2 - ivt}{1 + ivt} d\sigma_{2n}(s, t) \right| = 0$$

and

$$\lim_{y, v \rightarrow 0^+} \sup_{n \geq 1} \left| \int_{\mathbb{R}^2} \frac{(iy)(iv)\sqrt{1+s^2}\sqrt{1+t^2}}{(1+iys)(1+ivt)} d\rho_n(s, t) \right| = 0.$$

These uniform limits imply that $R_{\nu_{k_n, \mu_n}}(-iy, -iv)$ tends to zero uniformly in n as $y, v \rightarrow 0^+$, just as we expected. \square

Our results motivate the following definition.

Definition 3.7. A bi-free partial R -transform R_ν on a domain Ω is said to be *infinitely divisible* if for each positive integer $m \geq 2$, there exists a probability law μ_m on \mathbb{R}^2 such that $R_\nu = mR_{\mu_m}$ in Ω . In this case, the law ν is said to be *bi-freely infinitely divisible*.

Example 3.8. (Product of \boxplus -infinitely divisible laws) All point masses are clearly bi-freely infinitely divisible. A less trivial way of constructing a bi-freely infinitely divisible law is to form the product measure of two \boxplus -infinitely divisible laws. Thus, given two sets of Lévy parameters (γ_1, σ_1) and (γ_2, σ_2) , the definition (2.3) and the one-dimensional Lévy-Khintchine formula imply that the product measure

$$\nu = \nu_{\boxplus}^{\gamma_1, \sigma_1} \otimes \nu_{\boxplus}^{\gamma_2, \sigma_2}$$

on \mathbb{R}^2 has the bi-free R -transform

$$R_\nu(z, w) = z \left[\gamma_1 + \int_{\mathbb{R}} \frac{z+s}{1-zs} d\sigma_1(s) \right] + w \left[\gamma_2 + \int_{\mathbb{R}} \frac{w+t}{1-wt} d\sigma_2(t) \right]$$

for $(z, w) \in (\mathbb{C} \setminus \mathbb{R})^2$. It is obvious that R_ν/m is the bi-free R -transform of the corresponding product measure

$$\nu_{\boxplus}^{\gamma_1/m, \sigma_1/m} \otimes \nu_{\boxplus}^{\gamma_2/m, \sigma_2/m}$$

for all $m \geq 2$.

Theorem 3.6 shows that the class of limits for $k_n R_{\mu_n}$ is precisely the class of all infinitely divisible R -transforms. In classical probability, this is the content of Khintchine's result on the infinite divisibility in terms of Fourier transform.

Theorem 3.9. (Khintchine Type Characterization for Infinite Divisibility) *Let ν be a probability measure on \mathbb{R}^2 , and let R_ν be its R -transform defined on a domain Ω . The law ν is bi-freely infinitely divisible if and only if there exist probability measures μ_n on \mathbb{R}^2 and unbounded positive integers k_n such that $[\mu_n^{(j)}]^{\boxplus k_n} \Rightarrow \nu^{(j)}$ ($j = 1, 2$) and $R_\nu = \lim_{n \rightarrow \infty} k_n R_{\mu_n}$ in Ω .*

Proof. We have seen an explanation for the “only if” statement at the beginning of this section. As for the “if” part, we have to show that the limit $R_\nu/m = \lim_{n \rightarrow \infty} [k_n/m] R_{\mu_n}$ is a bi-free partial R -transform for each $m \geq 2$. This, however, is precisely the content of Theorem 3.6. \square

Therefore, Theorem 3.9 and Corollary 3.3 together provide a complete answer to Problem 3.1.

On the other hand, the proof of Theorem 3.6 shows that the infinitely divisible laws can be approximated by Poisson distributions. This perspective leads to another characterization of infinite divisibility in bi-free probability. To describe this result, we need a preparatory lemma.

Lemma 3.10. *Let τ and τ' be two finite positive Borel measures on \mathbb{R}^2 satisfying*

$$(3.6) \quad \frac{t^2}{1+t^2} d\tau(s, t) = \frac{t^2}{1+t^2} d\tau'(s, t)$$

on the Borel σ -field of \mathbb{R}^2 and having the same marginal

$$\tau \circ \pi_1^{-1} = \tau' \circ \pi_1^{-1}$$

with respect to the projection $\pi_1(s, t) = s$. Then we have $\tau = \tau'$.

Proof. It suffices to show that $\tau(E) = \tau'(E)$ for any open rectangle $E = I \times (a, b)$, where I and (a, b) are (bounded or unbounded) open intervals in \mathbb{R} . If $a > 0$ or $b < 0$, then the relation (3.6) shows that the desired identity holds. If $a = 0$, the continuity of τ and τ' yields that $\tau(E) = \lim_{\epsilon \rightarrow 0^+} \tau(I \times (\epsilon, b)) = \lim_{\epsilon \rightarrow 0^+} \tau'(I \times (\epsilon, b)) = \tau'(E)$. Similarly, the desired identity holds if $b = 0$. On the other hand, the assumption of admitting the same marginal yields that $\tau((a, b) \times \mathbb{R}) = \tau'((a, b) \times \mathbb{R})$, from which, along with the established result for the cases when $(a, b) = (0, \infty)$ and $(a, b) = (-\infty, 0)$, we see that $\tau((a, b) \times \{0\}) = \tau'((a, b) \times \{0\})$. In general, if $c < 0 < d$, then applying the above result to the sets $(a, b) \times (0, d)$, $(a, b) \times \{0\}$, and $(a, b) \times (c, 0)$ shows the desired result. This finishes the proof. \square

As seen in the proof of Theorem 3.6, the Poisson approximation $\nu_{k_n, \mu_n} \Rightarrow \nu$ holds along a subsequence of positive integers, thanks to the validity of the weak convergences

$$(3.7) \quad \frac{k_n s^2}{1+s^2} d\mu_n(s, t) \Rightarrow \tau_1, \quad \frac{k_n t^2}{1+t^2} d\mu_n(s, t) \Rightarrow \tau_2,$$

and

$$\frac{k_n s t}{\sqrt{1+s^2}\sqrt{1+t^2}} d\mu_n(s, t) \Rightarrow \rho$$

along the same subsequence. (Of course, the last weak convergence is in fact a genuine limit rather than a subsequential one.)

It is easy to verify that the weak limits τ_1 and ρ must satisfy the identity

$$\frac{t}{\sqrt{1+t^2}} d\tau_1(s, t) = \frac{s}{\sqrt{1+s^2}} d\rho(s, t)$$

on \mathbb{R}^2 . In particular, this shows that the relationship (3.6) must hold for any weak limit points τ_1 and τ'_1 of the sequence $k_n s^2 / (1 + s^2) d\mu_n$. Also, the measures τ_1 and τ'_1 have the same marginal law on the s -axis by the weak convergence (3.2). Thus, Lemma 3.10 implies that $\tau_1 = \tau'_1$. Similarly, it can be shown that the limit law τ_2 is also unique. Therefore, the weak convergences (3.7), as well as the Poisson approximation to ν , actually hold without passing to subsequences. Thus, we have obtained the following characterization for bi-free infinite divisibility.

Proposition 3.11. (Convergence of the accompanying Poisson laws) *Let ν be a probability measure on \mathbb{R}^2 . Then ν is bi-freely infinitely divisible if and only if there exist probability laws μ_n on \mathbb{R}^2 and unbounded positive integers k_n such that the bi-free compound Poisson laws ν_{k_n, μ_n} converge weakly to the measure ν on \mathbb{R}^2 .*

We would like to conclude this section by pointing out some interesting properties of \mathcal{BID} , the class of bi-freely infinitely divisible laws. Recall that the integral form of the R -transform of a measure $\nu \in \mathcal{BID}$ is given by

$$R_\nu(z, w) = zR_{\nu_{\boxplus}^{\gamma_1, \sigma_1}}(z) + wR_{\nu_{\boxplus}^{\gamma_2, \sigma_2}}(w) + G_{\sqrt{1+s^2}\sqrt{1+t^2} d\rho}(1/z, 1/w),$$

where $\nu_{\boxplus}^{\gamma_j, \sigma_j}$ ($j = 1, 2$) and ρ are the limit laws as in (3.2) and Theorem 3.2 (3). The quintuple $\Lambda(\nu) = (\gamma_1, \gamma_2, \sigma_1, \sigma_2, \rho)$ of limits is uniquely associated with the given infinitely divisible law ν , regardless of what approximants $\{\mu_n\}_{n=1}^\infty$ and $\{k_n\}_{n=1}^\infty$ may be used to obtain them. Being limits, such quintuples are closed under componentwise addition and multiplication by positive real numbers. This implies that the bi-free convolution \boxplus can be extended from compactly supported probabilities to bi-freely infinitely divisible laws; namely, for any $\nu_1, \nu_2 \in \mathcal{BID}$, their bi-free convolution $\nu_1 \boxplus \nu_2$ can be defined as the unique bi-freely infinitely divisible law satisfying

$$\Lambda(\nu_1 \boxplus \nu_2) = \Lambda(\nu_1) + \Lambda(\nu_2).$$

We record this finding formally as

Proposition 3.12. (Generalized bi-free convolution) *There exists an associative and commutative binary operation $\boxplus : \mathcal{BID} \times \mathcal{BID} \rightarrow \mathcal{BID}$ such that for any $\nu_1, \nu_2 \in \mathcal{BID}$, the relationship*

$$R_{\nu_1 \boxplus \nu_2} = R_{\nu_1} + R_{\nu_2}$$

holds in $(\mathbb{C} \setminus \mathbb{R})^2$.

In addition, the map Λ is injective and weakly continuous. The latter continuity means that if $\Lambda(\nu_n) = (\gamma_{1n}, \gamma_{2n}, \sigma_{1n}, \sigma_{2n}, \rho_n)$, $\Lambda(\nu) = (\gamma_1, \gamma_2, \sigma_1, \sigma_2, \rho)$, and $\nu_n \Rightarrow \nu$,

then we have $\gamma_{jn} \rightarrow \gamma_j$, $\sigma_{jn} \Rightarrow \sigma_j$, and $\rho_n \Rightarrow \rho$ as $n \rightarrow \infty$ for $j = 1, 2$. This can be easily verified using the free harmonic analysis results in [2] and Proposition 2.3.

However, the map Λ is not surjective, and its actual range will be studied in the next section.

Given a law $\nu \in \mathcal{BID}$, Theorem 3.6 shows that for each $t > 0$, there exists a unique law $\nu_t \in \mathcal{BID}$ such that

$$R_{\nu_t} = \lim_{n \rightarrow \infty} [tk_n] R_{\mu_n} = t R_\nu$$

in $(\mathbb{C} \setminus \mathbb{R})^2$, where $R_\nu = \lim_{n \rightarrow \infty} k_n R_{\mu_n}$ for some $\{\mu_n\}_{n=1}^\infty$ and $\{k_n\}_{n=1}^\infty$. As we mentioned earlier, this construction of ν_t is independent of the choice of the sequences $\{\mu_n\}_{n=1}^\infty$ and $\{k_n\}_{n=1}^\infty$. Thus, setting $\nu_0 = \delta_{(0,0)}$, we then have

$$\Lambda(\nu_t) = t \Lambda(\nu), \quad t \geq 0,$$

and the resulting family $\{\nu_t\}_{t \geq 0}$ forms a weakly continuous semigroup of probabilities on \mathbb{R}^2 under the (generalized) bi-free convolution \boxplus in the sense that

$$\nu_{s+t} = \nu_s \boxplus \nu_t, \quad s, t \geq 0,$$

and $\nu_t \Rightarrow \nu_0 = \delta_{(0,0)}$ as $t \rightarrow 0^+$. Conversely, if $\{\nu_t\}_{t \geq 0}$ is a given weakly continuous \boxplus -semigroup of probabilities on \mathbb{R}^2 , then each ν_t is bi-freely infinitely divisible. In particular, the law ν_1 belongs to the class \mathcal{BID} and generates the entire process $\{\nu_t\}_{t \geq 0}$ in terms of bi-free R -transform. Since this connection between infinitely divisible laws and continuous semigroups will play a role in the next section, we summarize our discussions into the following

Proposition 3.13. (Embedding Property) *Let ν be a probability measure on \mathbb{R}^2 . Then $\nu \in \mathcal{BID}$ if and only if there exists a weakly continuous \boxplus -semigroup $\{\nu_t\}_{t \geq 0}$ of probability measures on \mathbb{R}^2 satisfying $\nu_1 = \nu$ and $R_{\nu_t} = t R_\nu$ in $(\mathbb{C} \setminus \mathbb{R})^2$.*

Finally, note that the marginals $\{\nu_t^{(j)}\}_{t \geq 0}$ of a \boxplus -semigroup $\{\nu_t\}_{t \geq 0}$ are themselves a continuous semigroup relative to free convolution on \mathbb{R} , and we have

$$R_{\nu_t^{(j)}}(z) = t R_{\nu^{(j)}}(z), \quad z \notin \mathbb{R},$$

for their one-dimensional R -transforms. We recall from [2] that the dynamics of the process $\{\nu_t^{(j)}\}_{t \geq 0}$ is governed by the complex Burgers' type PDE:

$$\partial_t G_{\nu_t^{(j)}}(z) + R_{\nu_t^{(j)}} \left(G_{\nu_t^{(j)}}(z) \right) \partial_z G_{\nu_t^{(j)}}(z) = 0$$

in the upper half-plane for $t \in [0, \infty)$, where the time-derivative of $G_{\nu_t^{(j)}}(z)$ at $t = 0$ is equal to the one-dimensional R -transform $R_{\nu^{(j)}}(z)$.

4. BI-FREE CONVOLUTION SEMIGROUPS AND LÉVY-KHINTCHINE REPRESENTATIONS

In classical probability, finding the Lévy-Khintchine representation of a continuous convolution semigroup $\{\mu_t\}_{t \geq 0}$ of probabilities on \mathbb{R} is related to the study of its generating functional $A = \lim_{t \rightarrow 0^+} (\mu_t - \delta_0)/t$, which is defined at least on the dual group of $(\mathbb{R}, +)$ via the Fourier transform $\hat{\mu}_t = \exp(tA)$. Fix a law $\nu \in \mathcal{BID}$ and consider the \boxplus -semigroup $\{\nu_t\}_{t \geq 0}$ generated by ν . In what follows, we shall study the bi-free R -transform R_ν from an infinitesimal perspective and show that the map R_ν arises as the time-derivative of the Cauchy transform G_{ν_t} at time $t = 0$. This approach gives rise to a canonical integral representation for infinitely divisible R -transforms.

We start by assuming that ν is compactly supported, so that its R -transform R_ν can be written as an absolutely convergent power series with real coefficients

$$(4.1) \quad R_\nu(z, w) = \sum_{m, n \geq 0} \kappa_{m, n} z^m w^n$$

in some bidisk $\Omega = \{(z, w) \in \mathbb{C}^2 : |z|, |w| < r\}$. Note that $\kappa_{0,0} = 0$, $zR_{\nu^{(1)}}(z) = R_\nu(z, 0)$, and $wR_{\nu^{(2)}}(w) = R_\nu(0, w)$ by Lemma 2.4. So, both $R_{\nu^{(1)}}$ and $R_{\nu^{(2)}}$ also have power series expansions of their own. We conclude that the functions $R_{\nu^{(j)}}$ ($j = 1, 2$) and R_ν are all uniformly bounded in Ω , and hence the semigroup property shows further that $R_{\nu_t^{(j)}}$ ($j = 1, 2$) and R_{ν_t} all tend to zero uniformly in Ω as the time parameter $t \rightarrow 0^+$. This fact and the definition (2.3) of R -transform imply that there exists a cutoff constant $t_0 > 0$ such that the following identity

$$(4.2) \quad G\left(t, K_t^{(1)}(z), K_t^{(2)}(w)\right) = \frac{zw}{1 + tzR_{\nu^{(1)}}(z) + twR_{\nu^{(2)}}(w) - tR_\nu(z, w)}$$

holds for $0 \leq t \leq t_0$ and $(z, w) \in \Omega^* = \Omega \setminus \{(0, 0)\}$, using the notations

$$G(t, z, w) = G_{\nu_t}(z, w), \quad K_t^{(1)}(z) = tR_{\nu^{(1)}}(z) + 1/z, \quad K_t^{(2)}(w) = tR_{\nu^{(2)}}(w) + 1/w.$$

The formula (4.2) shows that the map $G\left(t, K_t^{(1)}(z), K_t^{(2)}(w)\right)$ is a C^1 -function in t and is holomorphic in (z, w) on the open set $(0, t_0) \times \Omega^*$.

Therefore, differentiating (4.2) at any $t > 0$ in the domain Ω^* yields

$$(4.3) \quad \begin{aligned} \partial_t G\left(t, K_t^{(1)}(z), K_t^{(2)}(w)\right) &= -R_{\nu^{(1)}}(z) \partial_z G\left(t, K_t^{(1)}(z), K_t^{(2)}(w)\right) \\ &\quad - R_{\nu^{(2)}}(w) \partial_w G\left(t, K_t^{(1)}(z), K_t^{(2)}(w)\right) \\ &\quad + \frac{zw(R_\nu(z, w) - zR_{\nu^{(1)}}(z) - wR_{\nu^{(2)}}(w))}{(1 + tzR_{\nu^{(1)}}(z) + twR_{\nu^{(2)}}(w) - tR_\nu(z, w))^2}. \end{aligned}$$

Meanwhile, the right-continuity $\nu_t \Rightarrow \delta_{(0,0)}$ and Proposition 2.2 imply that

$$\begin{aligned} \lim_{t \rightarrow 0^+} \partial_z G \left(t, K_t^{(1)}(z), K_t^{(2)}(w) \right) &= \lim_{t \rightarrow 0^+} \int_{\mathbb{R}^2} \frac{-1}{(K_t^{(1)}(z) - x)^2 (K_t^{(2)}(w) - y)} d\nu_t(x, y) \\ &= -z^2 w \end{aligned}$$

and

$$\lim_{t \rightarrow 0^+} \partial_w G \left(t, K_t^{(1)}(z), K_t^{(2)}(w) \right) = -zw^2.$$

Combining these facts with an application of the mean value theorem, we find that for any fixed $(z, w) \in \Omega^*$ the function

$$t \mapsto G \left(t, K_t^{(1)}(z), K_t^{(2)}(w) \right)$$

is also right-differentiable at $t = 0$ and this derivative can be further evaluated by taking $t \rightarrow 0^+$ in (4.3). Taking the fact $\lim_{t \rightarrow 0^+} K_t^{(j)}(\lambda) = 1/\lambda$ ($j = 1, 2$) into account, we then obtain the following result.

Proposition 4.1. *We have the right-derivative*

$$\lim_{t \rightarrow 0^+} \frac{G(t, 1/z, 1/w) - G(0, 1/z, 1/w)}{t} = zwR_\nu(z, w)$$

for $(z, w) \in \Omega^*$.

Thus, the distributional derivative $\lim_{t \rightarrow 0^+} (\nu_t - \nu_0)/t$ exists at the level of Cauchy transforms. To further analyze this derivative, we will require the tightness of the process ν_t in finite time. For this purpose, recall from [6] that the infinitely divisible measure ν can be realized as the joint distribution of the *commuting* bounded selfadjoint operators

$$a = \ell(f) + \ell(f)^* + \Lambda_{\text{left}}(T_1) + \kappa_{1,0}I \quad \text{and} \quad b = r(g) + r(g)^* + \Lambda_{\text{right}}(T_2) + \kappa_{0,1}I$$

on a certain full Fock space $\mathcal{F}(H)$ (with the vacuum state) by distinguishing the left action of the creation operator $\ell(f)$ and the gauge operator Λ_{left} from the right action of the operators $r(g)$ and Λ_{right} of the same nature. Here, the vectors f, g in the Hilbert space H and the commuting selfadjoint bounded operators T_1 and T_2 on H are chosen according to the distribution ν . Consider next the Hilbert space tensor product $L^2([0, \infty), dx) \otimes H$, then the realization of the process ν_t is given by the two-faced pair (a_t, b_t) of the form:

$$a_t = \ell(\chi_t \otimes f) + \ell(\chi_t \otimes f)^* + \Lambda_{\text{left}}(M_t \otimes T_1) + \kappa_{1,0}I$$

and

$$b_t = r(\chi_t \otimes g) + r(\chi_t \otimes g)^* + \Lambda_{\text{right}}(M_t \otimes T_2) + \kappa_{0,1}I,$$

where χ_t is the indicator function of the interval $[0, t)$ and M_t denotes the multiplication operator associated with the function χ_t on $L^2([0, \infty), dx)$. We refer the reader to [6] for the details of this construction; our point here, however, is that since the norms of a_t and b_t (and hence their spectral radii) are uniformly bounded when $0 \leq t \leq t_0$, we are able to conclude that the support of the process ν_t is uniformly bounded before the cutoff time t_0 (actually, within any finite time). In particular, the family $\{\nu_t : 0 \leq t \leq t_0\}$ is tight. We also mention that the real numbers $\kappa_{1,0}$ and $\kappa_{0,1}$ here are the first two coefficients in the power series expansion (4.1), and they represent the mean vector of ν .

We are now ready to present the bi-free Lévy-Khintchine representation for compactly supported, infinitely divisible laws.

Theorem 4.2. (Bi-free Lévy-Khintchine formula for compactly supported measures)
Let ν be a compactly supported measure in \mathcal{BID} , and let $\{\nu_t\}_{t \geq 0}$ be the \boxplus -semigroup generated by ν . Then there exists a unique triple (ρ_1, ρ_2, ρ) of two compactly supported positive Borel measures ρ_1 and ρ_2 and a compactly supported Borel signed measure ρ on \mathbb{R}^2 such that

(1) *The weak convergences*

$$\frac{s^2}{\varepsilon} d\nu_\varepsilon(s, t) \Rightarrow \rho_1, \quad \frac{t^2}{\varepsilon} d\nu_\varepsilon(s, t) \Rightarrow \rho_2, \quad \frac{st}{\varepsilon} d\nu_\varepsilon(s, t) \Rightarrow \rho$$

on \mathbb{R}^2 and the limits

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^2} s d\nu_\varepsilon(s, t) \rightarrow \kappa_{1,0} \quad \text{and} \quad \frac{1}{\varepsilon} \int_{\mathbb{R}^2} t d\nu_\varepsilon(s, t) \rightarrow \kappa_{0,1}$$

hold simultaneously as $\varepsilon \rightarrow 0^+$;

(2) *We have*

$$(4.4) \quad R_\nu(z, w) = \kappa_{1,0}z + \kappa_{0,1}w + \int_{\mathbb{R}^2} \frac{z^2}{1 - zs} d\rho_1(s, t) + \int_{\mathbb{R}^2} \frac{w^2}{1 - wt} d\rho_2(s, t) \\ + \int_{\mathbb{R}^2} \frac{zw}{(1 - zs)(1 - wt)} d\rho(s, t)$$

for (z, w) in $(\mathbb{C} \setminus \mathbb{R})^2 \cup \{(0, 0)\}$;

- (3) The total mass $\rho_j(\mathbb{R}^2)$ is equal to the variance of the marginal $\nu^{(j)}$ for $j = 1, 2$, the number $\rho(\mathbb{R}^2)$ is the covariance of ν , and the measures ρ_1 , ρ_2 , and ρ satisfy

$$(4.5) \quad \begin{cases} t d\rho_1 = s d\rho; \\ s d\rho_2 = t d\rho, \end{cases}$$

and

$$(4.6) \quad |\rho(\{(0, 0)\})|^2 \leq \rho_1(\{(0, 0)\})\rho_2(\{(0, 0)\}).$$

Proof. First, once we can show that the three families $\{(1/\varepsilon) s^2 d\nu_\varepsilon : 0 < \varepsilon \leq t_0\}$, $\{(1/\varepsilon) t^2 d\nu_\varepsilon : 0 < \varepsilon \leq t_0\}$, and $\{(1/\varepsilon) st d\nu_\varepsilon : 0 < \varepsilon \leq t_0\}$ are all bounded in total variation norm, the existence for compactly supported limit laws ρ_1 , ρ_2 , and ρ will become evident. This is because the family $\{\nu_\varepsilon : 0 \leq \varepsilon \leq t_0\}$ is tight and has a uniformly bounded support. Secondly, after the existence of the limits is established, we shall proceed to prove their uniqueness and the formulas (4.4)-(4.6).

We start with Proposition 4.1 which states that the limit

$$\begin{aligned} zwR_\nu(z, w) &= \lim_{\varepsilon \rightarrow 0^+} (1/\varepsilon) [G(\varepsilon, 1/z, 1/w) - G(0, 1/z, 1/w)] \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \left[\int_{\mathbb{R}^2} \frac{zw}{(1-zs)(1-wt)} d\nu_\varepsilon(s, t) - zw \right] \\ &= zw \left[\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left(\frac{zs}{1-zs} + \frac{wt}{1-wt} + \frac{zwt}{(1-zs)(1-wt)} \right) d\nu_\varepsilon(s, t) \right] \end{aligned}$$

holds for (z, w) in the punctured bidisk Ω^* . Hence, we have the identity

$$(4.7) \quad R_\nu(z, w) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \left[\frac{zs}{1-zs} + \frac{wt}{1-wt} + \frac{zwt}{(1-zs)(1-wt)} \right] d\nu_\varepsilon(s, t)$$

in the bidisk Ω , for the integrand is equal to zero for any (s, t) if $z = 0 = w$.

So, by letting $w = 0$ in (4.7), Lemma 2.4 implies

$$zR_{\nu^{(1)}}(z) = R_\nu(z, 0) = z \left[\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \frac{s}{1-zs} d\nu_\varepsilon(s, t) \right]$$

or, equivalently,

$$(4.8) \quad R_{\nu^{(1)}}(z) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \frac{s}{1-zs} d\nu_\varepsilon(s, t)$$

for $|z| < r$. After plugging $z = 0$ in this formula, we obtain the limiting formula for the constant $\kappa_{1,0}$. Moreover, by taking the imaginary part of (4.8), we reach

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^2} \frac{\Im z}{|1-zs|^2} \frac{s^2}{\varepsilon} d\nu_\varepsilon(s, t) = \Im R_{\nu^{(1)}}(z), \quad |z| < r.$$

This shows that the family $\{(1/\varepsilon) s^2 d\nu_\varepsilon : 0 < \varepsilon \leq t_0\}$ has uniformly bounded total variation norms, because the integrand $\Im z/|1 - zs|^2$ is uniformly bounded away from zero and from infinity for s in the uniform support of $\{\nu_\varepsilon : 0 \leq \varepsilon \leq t_0\}$ and for $r/2 < |z| < r$, $z \notin \mathbb{R}$. We deduce from the same kind of argument that

$$(4.9) \quad R_{\nu^{(2)}}(w) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \frac{t}{1 - wt} d\nu_\varepsilon(s, t), \quad |w| < r,$$

$$\kappa_{0,1} = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^2} t d\nu_\varepsilon(s, t),$$

and that the set $\{(1/\varepsilon) t^2 d\nu_\varepsilon : 0 < \varepsilon \leq t_0\}$ is bounded in total variation norm. Finally, the boundedness of the remaining family $\{(1/\varepsilon) st d\nu_\varepsilon : 0 < \varepsilon \leq t_0\}$ is an easy consequence of the Cauchy-Schwarz inequality. So, the existence of the limit laws is proved.

In view of (4.8) and (4.9), we can now split (4.7) into three limits, replace the first two with the corresponding R -transforms, and finally get

$$(4.10) \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^2} \frac{zw}{(1 - zs)(1 - wt)} \frac{st}{\varepsilon} d\nu_\varepsilon(s, t) = R_\nu(z, w) - zR_{\nu^{(1)}}(z) - wR_{\nu^{(2)}}(w)$$

for $(z, w) \in \Omega$. This shows that if ρ and ρ' are two weak limits of the signed measures $\{(1/\varepsilon) st d\nu_\varepsilon : 0 < \varepsilon \leq t_0\}$ as $\varepsilon \rightarrow 0^+$, then they must satisfy

$$G_\rho(1/z, 1/w) = G_{\rho'}(1/z, 1/w)$$

for (z, w) in the open set Ω^* (and hence everywhere by analytic extension), proving that $\rho = \rho'$. Therefore, the limit law ρ is unique and $(1/\varepsilon) st d\nu_\varepsilon \Rightarrow \rho$. In addition, it follows from (4.10) and the power series expansion (4.1) that

$$\rho(\mathbb{R}^2) = \kappa_{1,1} = \text{Covariance}(\nu).$$

To see the uniqueness of the other two limit laws ρ_1 and ρ_2 , we first expand the integrand in (4.8) into a power series of z and then use the fact that any limit ρ_1 of $\{(1/\varepsilon) s^2 d\nu_\varepsilon : 0 < \varepsilon \leq t_0\}$ is compactly supported to get

$$(4.11) \quad R_{\nu^{(1)}}(z) = \kappa_{1,0} + \sum_{m=0}^{\infty} \left(\int_{\mathbb{R}^2} s^m d\rho_1(s, t) \right) z^{m+1}, \quad |z| < r.$$

Similarly, for $|w| < r$, we have

$$(4.12) \quad R_{\nu^{(2)}}(w) = \kappa_{0,1} + \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}^2} t^n d\rho_2(s, t) \right) w^{n+1}.$$

On the other hand, we should do the same for (4.10); only this time we will use the following decomposition

$$\frac{st}{(1-zs)(1-wt)} = st + \frac{ws}{1-wt} \cdot t^2 + \frac{zt}{(1-zs)(1-wt)} \cdot s^2$$

to obtain

$$\begin{aligned} R_\nu(z, w) - zR_{\nu^{(1)}}(z) - wR_{\nu^{(2)}}(w) &= \kappa_{1,1}zw + \sum_{n \geq 0} \left(\int_{\mathbb{R}^2} st^n d\rho_2(s, t) \right) zw^{n+2} \\ &\quad + \sum_{m, n \geq 0} \left(\int_{\mathbb{R}^2} s^m t^{n+1} d\rho_1(s, t) \right) z^{m+2} w^{n+1}. \end{aligned}$$

Combining this with (4.11), (4.12), and the original power series expansion (4.1) of R_ν , we have shown the following identity

$$\begin{aligned} \sum_{m, n \geq 0} \kappa_{m, n} z^m w^n &= \kappa_{1,0}z + \kappa_{0,1}w + \sum_{m=0}^{\infty} M_{m,0}(\rho_1) z^{m+2} + \sum_{n=0}^{\infty} M_{0,n}(\rho_2) w^{n+2} \\ &\quad + \kappa_{1,1}zw + \sum_{n \geq 0} M_{1,n}(\rho_2) zw^{n+2} + \sum_{m, n \geq 0} M_{m,n+1}(\rho_1) z^{m+2} w^{n+1} \end{aligned}$$

of power series in the open set Ω , where the notation

$$M_{m,n}(\rho_j) = \int_{\mathbb{R}^2} s^m t^n d\rho_j(s, t), \quad m, n \geq 0, \quad j = 1, 2.$$

Since all these power series converge absolutely, we are allowed to rearrange the order of summation freely. By the uniqueness of power series expansion in open sets, we conclude that the moments of the limiting measure ρ_1 (and hence ρ_1 itself) are uniquely determined by the given sequence $\{\kappa_{m,n}\}_{m,n \geq 0}$ of coefficients and therefore the weak convergence $(1/\varepsilon) s^2 d\nu_\varepsilon \Rightarrow \rho_1$ holds. Also, by comparing the coefficients in the preceding identity of power series, we have

$$\rho_1(\mathbb{R}^2) = \kappa_{2,0} = \text{Variance}(\nu^{(1)}).$$

The uniqueness of ρ_2 and its statistics can be shown in the same way, and we shall not repeat this argument. Thus, the statement (1) of the theorem is proved.

The integral representation (4.4) is a direct consequence of (4.7) and the convergences in (1).

To show the system (4.5), take any continuous and bounded function φ on \mathbb{R}^2 , the weak convergence results in (1) imply

$$\int_{\mathbb{R}^2} \varphi(s, t) t d\rho_1(s, t) = \lim_{\varepsilon \rightarrow 0^+} (1/\varepsilon) \int_{\mathbb{R}^2} \varphi(s, t) t s^2 d\nu_\varepsilon(s, t) = \int_{\mathbb{R}^2} \varphi(s, t) s d\rho(s, t),$$

from which we deduce that $t d\rho_1(s, t) = s d\rho(s, t)$. Similarly, $s d\rho_2(s, t) = t d\rho(s, t)$ holds.

Finally, for the inequality (4.6), let φ_n be a sequence of continuous functions such that $0 \leq \varphi_n(s, t) \leq 1$ and $\lim_{n \rightarrow \infty} \varphi_n(s, t) = I_{\{(0,0)\}}(s, t)$, the indicator function of the singleton $\{(0,0)\}$, for all $(s, t) \in \mathbb{R}^2$. An application of the Cauchy-Schwarz inequality to the measure $(1/\varepsilon)\varphi_n d\nu_\varepsilon$ shows that

$$\left| \int_{\mathbb{R}^2} \varphi_n(s, t) \frac{st}{\varepsilon} d\nu_\varepsilon(s, t) \right|^2 \leq \int_{\mathbb{R}^2} \varphi_n(s, t) \frac{s^2}{\varepsilon} d\nu_\varepsilon(s, t) \int_{\mathbb{R}^2} \varphi_n(s, t) \frac{t^2}{\varepsilon} d\nu_\varepsilon(s, t)$$

for all $\varepsilon > 0$. By letting $\varepsilon \rightarrow 0^+$, we obtain the estimate

$$\left| \int_{\mathbb{R}^2} \varphi_n(s, t) d\rho(s, t) \right|^2 \leq \int_{\mathbb{R}^2} \varphi_n(s, t) d\rho_1(s, t) \int_{\mathbb{R}^2} \varphi_n(s, t) d\rho_2(s, t), \quad n \geq 1.$$

The inequality (4.6) follows from this estimate and the dominated convergence theorem. The theorem is now completely proved. \square

We mention that the integral formula (4.4) for compactly supported, bi-freely infinitely divisible measures was first obtained in [6] by means of combinatorial and operator-theoretical methods.

We next present our main result in which the boundedness condition for the support of ν is no longer needed. Note that the following result is stronger than the one obtained in [6], because it provides a complete parametrization for the entire class \mathcal{BID} .

Theorem 4.3. (General bi-free Lévy-Khintchine representation) *Let R be a given holomorphic function defined on the product domain $\Omega = (\Delta \cup \overline{\Delta}) \times (\Delta \cup \overline{\Delta})$ associated with some Stolz angle Δ . Then the following statements are equivalent:*

- (1) *There exists a law $\nu \in \mathcal{BID}$ such that $R = R_\nu$ on Ω .*
- (2) *There exist $\gamma_1, \gamma_2 \in \mathbb{R}$, two finite Borel positive measures ρ_1 and ρ_2 on \mathbb{R}^2 , and a finite Borel signed measure ρ on \mathbb{R}^2 such that*

$$(4.13) \quad \begin{cases} t/\sqrt{1+t^2} d\rho_1 = s/\sqrt{1+s^2} d\rho; \\ s/\sqrt{1+s^2} d\rho_2 = t/\sqrt{1+t^2} d\rho; \\ |\rho(\{(0,0)\})|^2 \leq \rho_1(\{(0,0)\})\rho_2(\{(0,0)\}), \end{cases}$$

and the function R extends analytically to $(\mathbb{C} \setminus \mathbb{R})^2$ via the formula:

$$(4.14) \quad \begin{aligned} R(z, w) = & \gamma_1 z + \gamma_2 w + \int_{\mathbb{R}^2} \frac{z^2 + zs}{1 - zs} d\rho_1(s, t) + \int_{\mathbb{R}^2} \frac{w^2 + wt}{1 - wt} d\rho_2(s, t) \\ & + \int_{\mathbb{R}^2} \frac{zw\sqrt{1+s^2}\sqrt{1+t^2}}{(1 - zs)(1 - wt)} d\rho(s, t). \end{aligned}$$

In this case, the quintuple $(\gamma_1, \gamma_2, \rho_1, \rho_2, \rho)$ is unique, and we have the marginal law $\nu^{(j)} = \nu_{\boxplus}^{\gamma_j, \rho_j^{(j)}}$ for $j = 1, 2$.

Proof. It is clear that only the implication from (2) to (1) needs a proof. Suppose we are given the integral form (4.14), whose representing measures ρ_1 , ρ_2 , and ρ satisfy the system (4.13). Let $S = \{(s, 0) \in \mathbb{R}^2 : s \neq 0\}$ and $T = \{(0, t) \in \mathbb{R}^2 : t \neq 0\}$ be the two punctured coordinate axes on the plane and let $U = \mathbb{R}^2 \setminus (S \cup T \cup \{(0, 0)\})$ be the slit plane. The sets S , T , and U are Borel measurable, and we can consider the following decompositions

$$\begin{cases} \rho_j = \rho_j(\{(0, 0)\})\delta_{(0,0)} + \rho_j^S + \rho_j^T + \rho_j^U, & j = 1, 2, \\ \rho = \rho(\{(0, 0)\})\delta_{(0,0)} + \rho^S + \rho^T + \rho^U. \end{cases}$$

A notation like ρ^S here means the restriction of the measure ρ on the Borel set S , i.e.,

$$\rho^S(E) = \rho(E \cap S)$$

for all Borel sets $E \subset \mathbb{R}^2$. We shall identify the measures restricted on S or T as measures on \mathbb{R} and write, with a slight abuse of notation, that $\rho_j^S = d\rho_j^S(s)$ and $\rho_j^T = d\rho_j^T(t)$.

To each $n \geq 1$, we introduce the set $T_n = \{(0, t) \in \mathbb{R}^2 : |t| \geq 1/n\}$ and observe from the dominated convergence theorem and the system (4.13) that

$$\begin{aligned} \rho_1(T) &= \lim_{n \rightarrow \infty} \rho_1(T_n) = \lim_{n \rightarrow \infty} \int_{T_n} \frac{\sqrt{1+t^2}}{t} \frac{t}{\sqrt{1+t^2}} d\rho_1(s, t) \\ &= \lim_{n \rightarrow \infty} \int_{T_n} \frac{\sqrt{1+t^2}}{t} \frac{s}{\sqrt{1+s^2}} d\rho(s, t) = \lim_{n \rightarrow \infty} \int_{T_n} 0 d\rho(s, t) = 0. \end{aligned}$$

This implies that the restricted measure ρ_1^T is in fact the zero measure. Similarly, one can show that the measures ρ_2^S , ρ^S , and ρ^T are also equal to the zero measure. Accordingly, the integral form (4.14) can be decomposed into

$$\begin{aligned} R(z, w) &= \rho_1(\{(0, 0)\})z^2 + \rho_2(\{(0, 0)\})w^2 + \rho(\{(0, 0)\})zw \\ &\quad + z \left[\gamma_1 + \int_{\mathbb{R} \setminus \{0\}} \frac{z+s}{1-zs} d\rho_1^S(s) \right] + w \left[\gamma_2 + \int_{\mathbb{R} \setminus \{0\}} \frac{w+t}{1-wt} d\rho_2^T(t) \right] \\ &\quad + \int_U \frac{z^2 + zs}{1-zs} d\rho_1^U(s, t) + \int_U \frac{w^2 + wt}{1-wt} d\rho_2^U(s, t) \\ &\quad + \int_U \frac{zw\sqrt{1+s^2}\sqrt{1+t^2}}{(1-zs)(1-wt)} d\rho^U(s, t), \end{aligned}$$

in which we find that

$$\rho_1(\{(0,0)\})z^2 + \rho_2(\{(0,0)\})w^2 + \rho(\{(0,0)\})zw = R_{\mu_1}(z, w)$$

for some bi-free Gaussian law μ_1 by (4.13) and that

$$z \left[\gamma_1 + \int_{\mathbb{R} \setminus \{0\}} \frac{z+s}{1-zs} d\rho_1^S(s) \right] + w \left[\gamma_2 + \int_{\mathbb{R} \setminus \{0\}} \frac{w+t}{1-wt} d\rho_2^T(t) \right]$$

is the bi-free R -transform of the product measure

$$\mu_2 = \nu_{\boxplus}^{\gamma_1, \rho_1^S} \otimes \nu_{\boxplus}^{\gamma_2, \rho_2^T}.$$

Note that both μ_1 and μ_2 are bi-freely infinitely divisible.

We shall argue that the remaining integral form $R_3 = R - R_{\mu_1} - R_{\mu_2}$ is also an infinitely divisible bi-free R -transform. Toward this end we consider the truncations

$$\rho_{jn}^U = \varphi_n \rho_j^U \quad \text{and} \quad \rho_n^U = \varphi_n \rho^U \quad (j = 1, 2),$$

where $0 \leq \varphi_n \leq 1$ is a continuous function on \mathbb{R}^2 such that $\varphi_n(s, t) = 1$ for $(s, t) \in U_n = \{(s, t) \in \mathbb{R}^2 : |s| \geq 1/n, |t| \geq 1/n\}$ and $\varphi_n(s, t) = 0$ on the complement $\mathbb{R}^2 \setminus U_{n+1}$. Clearly, we have $\rho_{jn}^U \Rightarrow \rho_j^U$ ($j = 1, 2$) and $\rho_n^U \Rightarrow \rho^U$ as $n \rightarrow \infty$, so that the corresponding sequence

$$(4.15) \quad R_n(z, w) = \int_{U_{n+1}} \frac{z^2 + zs}{1-zs} d\rho_{1n}^U(s, t) + \int_{U_{n+1}} \frac{w^2 + wt}{1-wt} d\rho_{2n}^U(s, t) \\ + \int_{U_{n+1}} \frac{zw\sqrt{1+s^2}\sqrt{1+t^2}}{(1-zs)(1-wt)} d\rho_n^U(s, t)$$

tends to $R_3(z, w)$ for each $z, w \notin \mathbb{R}$. In addition, it is easy to see that the limit

$$\lim_{y, v \rightarrow 0^+} R_n(-iy, -iv) = 0$$

holds uniformly for all n .

Therefore, if we can show that each R_n is an infinitely divisible bi-free R -transform, then Proposition 2.6 and the fact that the family \mathcal{BID} is closed under the topology of weak convergence would imply that $R_3 = R_{\mu_3}$ for some $\mu_3 \in \mathcal{BID}$. In that way, the desired probability measure ν could be given by the generalized bi-free convolution

$$\nu = \mu_1 \boxplus \boxplus \mu_2 \boxplus \boxplus \mu_3$$

among these infinitely divisible laws. For this purpose, we fix n and introduce a finite Borel signed measure τ on U_{n+1} by

$$(4.16) \quad \tau = \frac{\sqrt{1+s^2}\sqrt{1+t^2}}{st} d\rho_n^U.$$

The conditions (4.13) imply that the measure

$$(4.17) \quad \tau = \frac{1+s^2}{s^2} d\rho_{1n}^U = \frac{1+t^2}{t^2} d\rho_{2n}^U,$$

and hence it is in fact a positive measure on the set U_{n+1} . If $\lambda = \tau(U_{n+1}) = 0$, then (4.16) and (4.17) imply that the measures ρ_{1n}^U , ρ_{2n}^U , and ρ_n^U are all equal to the zero measure. So, the function R_n in this case is constantly zero, and hence it is the infinitely divisible R -transform corresponding to the point mass at $(0, 0)$. Therefore, we assume $\lambda > 0$ in the sequel and normalize the measure τ to get the probability law

$$\mu = \tau/\lambda.$$

Finally, define the constants

$$a = -\lambda \int_{U_{n+1}} \frac{s}{1+s^2} d\mu(s, t) \quad \text{and} \quad b = -\lambda \int_{U_{n+1}} \frac{t}{1+t^2} d\mu(s, t).$$

We now combine (4.15), (4.16), and (4.17) with the identity

$$\frac{z+x}{1-zx} \frac{x^2}{1+x^2} = \frac{x}{1-zx} - \frac{x}{1+x^2}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad x \in \mathbb{R},$$

to get

$$R_n(z, w) = az + bw - \lambda + \lambda \int_{U_{n+1}} \frac{1}{(1-zs)(1-wt)} d\mu(s, t).$$

This proves that R_n is the R -transform of the bi-free convolution

$$\delta_{(a,b)} \boxplus \boxplus \nu_{\lambda,\mu},$$

which is indeed infinitely divisible. The proof is now completed. \square

The attentive reader may notice that the integral representation (4.14) could have been derived directly from the general limit theorems in Section 3 through a discretization process of the \boxplus -semigroup $\{\nu_t\}_{t \geq 0}$. However, the approach undertaken here has the advantage that not only does it reveal how the conditions (4.13) arise naturally from the dynamical view of R -transform (hence justifying the name "Lévy-Khintchine formula"), but it also demonstrates that at any time t , the process ν_t can be realized as

$$\nu_t = \text{Gaussian} \boxplus \boxplus \text{1-D infinitely divisible product} \boxplus \boxplus \text{Poisson limit},$$

which resembles its Fock space model in the bounded support case.

Finally, we remark that the integral formulas (4.4) and (4.14) are equivalent when the infinitely divisible law ν is compactly supported. This follows from an easy substitution: $d\rho'_1 = (1 + s^2) d\rho_1$, $d\rho'_2 = (1 + t^2) d\rho_2$, $d\rho' = \sqrt{1 + s^2}\sqrt{1 + t^2} d\rho$, and

$$a_1 = \gamma_1 + \int_{\mathbb{R}^2} s d\rho_1(s, t), \quad a_2 = \gamma_2 + \int_{\mathbb{R}^2} t d\rho_2(s, t),$$

which turns the integral form (4.14) into

$$\begin{aligned} R_\nu(z, w) &= a_1 z + a_2 w + \int_{\mathbb{R}^2} \frac{z^2}{1 - zs} d\rho'_1(s, t) + \int_{\mathbb{R}^2} \frac{w^2}{1 - wt} d\rho'_2(s, t) \\ &\quad + \int_{\mathbb{R}^2} \frac{zw}{(1 - zs)(1 - wt)} d\rho'(s, t), \end{aligned}$$

and the system (4.13) now becomes

$$\begin{cases} t d\rho'_1 = s d\rho'; \\ s d\rho'_2 = t d\rho'; \\ |\rho'(\{(0, 0)\})|^2 \leq \rho'_1(\{(0, 0)\})\rho'_2(\{(0, 0)\}). \end{cases}$$

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